

On a matrix method for the study of small perturbations in galaxies

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ABSTRACT

A matrix method is formulated in a Lagrangian representation for the solution of the characteristic value problem governing modes of oscillation and instability in a collisionless stellar system. The underlying perturbation equations govern the Lagrangian displacement of a star in the six-dimensional phase space. This matrix method has a basis in a variational principle. The method is developed in detail for radial oscillations of a spherical system. The basis vectors required for the representation of the Lagrangian displacement in this case are derived from solutions of the Lagrangian perturbation equations for radial perturbations in a homogeneous sphere. The basis vectors are made divergence free in the six-dimensional phase space in accordance with the requirement of Liouville's theorem that the flow of the system in the phase space must be incompressible. The basis vectors are made orthogonal with respect to a properly chosen set of adjoint vectors with the aid of a Gram-Schmidt procedure. Some basis vectors are null vectors in the sense that their inner products with their own adjoint vectors vanish. The characteristic frequencies of the lowest radial modes are calculated in several approximations for members of a family of spherical models which span a wide range of central concentrations. The present formulation of the matrix method can be generalized for nonradial modes in spherical systems and for modes in axisymmetric systems.

Key words: instabilities – galaxies: kinematics and dynamics: structure.

1 INTRODUCTION

Considerations of stability and instability impose certain constraints on the equilibrium and evolution of a galaxy. Most galaxies observed in nature are presumed to be in stable states of equilibrium, or nearly so, inasmuch as they would not be expected to survive in unstable states long enough to be observed. If the processes of formation or the subsequent processes of evolution bring a galaxy into an unstable state, then the onset and development of that instability would act as dynamical mechanisms of evolution. Consequences of the instability could be manifest in a later equilibrium state of such a galaxy.

Examples of instabilities in galaxies are well known. According to Toomre (1964), an axisymmetric disk would be locally unstable with respect to axisymmetric perturbations unless the dispersion of the peculiar velocities of the stars exceeded a certain critical value. The conjecture of Ostriker & Peebles (1973) that excessive rotation would make an axisymmetric galaxy unstable with respect to a deformation into a bar has become a central organizing principle in efforts to understand the dynamics of spiral galaxies. A third example, originally suggested by Antonov (1973) and demonstrated for the first time by Polyachenko & Shukhman (1981) and Polyachenko (1981), is the radial orbit instability in which the anisotropy of the velocity distribution makes a spherically symmetric stellar system unstable with respect to an axisymmetric deformation. Finally, certain firehose instabilities can deform the equatorial plane of an axisymmetric stellar system if the system is sufficiently flattened, and those instabilities might, according to Fridman & Polyachenko (1984; see Chapter X), explain the absence of elliptical galaxies flatter than type E7 on the Hubble sequence.

It is beyond the scope of this introduction to present an extensive review of the history of investigations of such instabilities. An introduction to the subject is contained in the textbook of Binney & Tremaine (1987), and the subject is treated extensively in the monographs of Fridman & Polyachenko (1984) and Palmer (1994). Reviews by Merritt (1987, 1990) and Polyachenko (1987) are useful supplements to the monographs. Accounts of subsequent investigations (e.g., Merritt & Sellwood 1994; Sellwood & Merritt 1994; Robijn 1995; Sellwood & Valluri 1997) include references to more recent developments.

For the work described in this paper, the most relevant part of the literature on small perturbations in galaxies concerns the matrix method formulated by Kalnajs (1977) for the solution of the characteristic value problem governing normal modes of oscillation and instability in a collisionless stellar system. Originally formulated for the study of instabilities in axisymmetric disks, that method has subsequently been used in several investigations of the radial orbit instability in spherical systems

(Polyachenko & Shukhman 1981; Fridman & Polyachenko 1984; Palmer & Papaloizou 1987; Weinberg 1989, 1991a; Saha 1991, Bertin et al. 1994). More recently, the method has been extended to the investigation of small perturbations in axisymmetric systems (Robijn 1995). The matrix method of Kalnajs is based on an Eulerian representation of the perturbations.

A Lagrangian representation of the perturbations provides an alternative framework for the investigation of the oscillations and the stability of stellar systems (Vandervoort 1983, 1989, 1991). This paper describes a matrix method for the study of normal modes in the Lagrangian representation. The elements of Lagrangian perturbation theory that provide the foundation for the method are reviewed in Section 2, and the method is formulated in Section 3. For the sake of definiteness and simplicity in this first presentation of the new matrix method, we concentrate on an application of the method to the study of radial modes of oscillation in a spherically symmetric galaxy. In Section 4 a procedure is described for the construction of basis vectors which can be used for the representation of such perturbations. The reduction and solution of the characteristic value problem in the matrix representation is described in Section 5, and Section 6 contains examples of solutions for modes which illustrate and test the method. A connection between the discrete and continuous spectra of modes in a stellar system is described in Section 7. Important technical details are described in a series of appendices.

As this paper was being written, substantial progress was made in the generalization of the present formulation of the matrix method for a wider range of applications. The most important details of such generalizations have already been worked out for non-radial modes of oscillation in spherical systems and for general oscillations of axisymmetric systems. The construction of basis vectors for the representation of non-radial modes in spheres follows the procedure described in Section 4 very closely. It consequently appears that much of the analysis described in this paper for radial oscillations in spheres can be readily extended to the study of non-radial oscillations in spheres. A similar construction of basis vectors for the representation of perturbations in axisymmetric systems can be achieved with the aid of a formulation based on an earlier investigation of the lowest modes of oscillation of a homogeneous spheroid (Vandervoort 1991). Finally, preliminary investigation suggests that this approach to the construction of basis vectors can be extended to nonrotating triaxial systems. The author plans next to apply these generalizations, in a continuation of the present work, to the radial orbit instability in spherical systems and to firehose instabilities in rotating axisymmetric systems.

2 LAGRANGIAN PERTURBATION THEORY

Small perturbations of a collisionless stellar system are described in the Lagrangian representation in terms of the Lagrangian displacement $(\Delta\mathbf{x}, \Delta\mathbf{v})$ of a star (Vandervoort 1983). Thus, if the position and velocity of the star at time t would have been \mathbf{x} and \mathbf{v} , respectively, in the unperturbed system, then the position and velocity of the star at that time are, by definition, $\mathbf{x} + \Delta\mathbf{x}$ and $\mathbf{v} + \Delta\mathbf{v}$, respectively, in the perturbed system.

For an infinitesimal perturbation, $\Delta\mathbf{x}$ and $\Delta\mathbf{v}$ are considered to be functions of \mathbf{x} , \mathbf{v} , and t , and the equations of motion for the perturbation are

$$\frac{d\Delta\mathbf{x}}{dt} = \Delta\mathbf{v} \quad \text{and} \quad \frac{d\Delta\mathbf{v}}{dt} = \Delta\mathbf{a}\{\Delta\mathbf{x}\}, \quad (1)$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} - \frac{\partial V_0}{\partial \mathbf{x}} \cdot \frac{\partial}{\partial \mathbf{v}} \quad (2)$$

is the total time derivative along the unperturbed trajectory of the star, and

$$\Delta\mathbf{a}\{\Delta\mathbf{x}\} = - \left(\Delta\mathbf{x} \cdot \frac{\partial}{\partial \mathbf{x}} \right) \frac{\partial V_0}{\partial \mathbf{x}} - \frac{\partial}{\partial \mathbf{x}} \left[Gm_* \frac{\partial}{\partial \mathbf{x}} \cdot \int_{\Omega} \frac{\Delta\mathbf{x}(\mathbf{x}', \mathbf{v}', t) f_0(\mathbf{x}', \mathbf{v}')}{|\mathbf{x}' - \mathbf{x}|} d\mathbf{x}' d\mathbf{v}' \right] \quad (3)$$

is the Lagrangian perturbation of the acceleration of the star. Here the distribution function $f_0(\mathbf{x}, \mathbf{v})$ represents the unperturbed density of stars in the six-dimensional phase space of a single star, and $V_0(\mathbf{x})$ represents the gravitational potential in the unperturbed system. Also, G denotes the constant of gravitation, and m_* denotes the mass of a single star. The second term on the right-hand side of equation (3) represents the Eulerian perturbation of the gravitational acceleration of a star, and the integration there extends over the region Ω of the phase space that is accessible to stars in the unperturbed system.

Equations (1)-(3) are a homogeneous system of linear integro-differential equations governing an infinitesimal perturbations of a stellar system. In Vandervoort (1989, hereafter LM), these equations have been reduced to a characteristic-value problem governing the normal modes of oscillation and instability of a system. The subsections that follow contain a review of the results derived in LM that are essential for the present formulation of a matrix method.

2.1 The characteristic value problem

For normal modes of oscillation having a time dependence of the form $\exp(-i\omega t)$, where the characteristic frequency ω is a constant, we represent the Lagrangian displacement as a six-vector

$$\boldsymbol{\eta} = \begin{pmatrix} \Delta\mathbf{x} \\ \Delta\mathbf{v} \end{pmatrix} \quad (4)$$

in the six-dimensional phase space, and we reduce equations (1) to the form

$$\omega \boldsymbol{\eta} = P \boldsymbol{\eta}. \quad (5)$$

Here P is the matrix of operators

$$P = \begin{pmatrix} -D & \mathbf{i} \\ \mathbf{i} \Delta \mathbf{a} \{\} & -D \end{pmatrix}, \quad (6)$$

where

$$D = \mathbf{i} \left(\mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} - \frac{\partial V_0}{\partial \mathbf{x}} \cdot \frac{\partial}{\partial \mathbf{v}} \right). \quad (7)$$

We identify $\boldsymbol{\eta}$ as an element of a Hilbert space defined by letting the inner product of two six-vectors

$$\mathbf{g}(\mathbf{x}, \mathbf{v}) = \begin{pmatrix} \mathbf{g}_1(\mathbf{x}, \mathbf{v}) \\ \mathbf{g}_2(\mathbf{x}, \mathbf{v}) \end{pmatrix} \quad \text{and} \quad \mathbf{h}(\mathbf{x}, \mathbf{v}) = \begin{pmatrix} \mathbf{h}_1(\mathbf{x}, \mathbf{v}) \\ \mathbf{h}_2(\mathbf{x}, \mathbf{v}) \end{pmatrix}, \quad (8)$$

say, where \mathbf{g}_1 , \mathbf{g}_2 , \mathbf{h}_1 , and \mathbf{h}_2 are three-vectors in the six-dimensional phase space, be

$$\langle \mathbf{g}, \mathbf{h} \rangle_2 = m_* \int_{\Omega} \mathbf{g}^*(\mathbf{x}, \mathbf{v}) \cdot \mathbf{h}(\mathbf{x}, \mathbf{v}) f_0(\mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v}, \quad (9)$$

where the asterisk, written as a superscript, signifies complex conjugation here and in what follows. In the notation adopted in LM, the subscript 2 appears on the left-hand side of equation (9) in order to distinguish the inner products of six-vectors from the inner products of three-vectors. The present definition of the inner product of two six-vectors differs slightly from the definition in LM; we now include a factor m_* which was previously omitted. It is to be emphasized here that $\mathbf{g}^* \cdot \mathbf{h}$ is the scalar product of two six-vector functions in the six-dimensional phase space.

It is shown in the Appendix of LM that the Lagrangian displacement satisfies a divergence condition of the form

$$\nabla_6 \cdot \boldsymbol{\eta} \equiv \frac{\partial}{\partial \mathbf{x}} \cdot \Delta \mathbf{x} + \frac{\partial}{\partial \mathbf{v}} \cdot \Delta \mathbf{v} = 0. \quad (10)$$

This condition implies, in accordance with Liouville's theorem, that the perturbed flow of the stars in the phase space is incompressible.

2.2 The adjoint problem

The operator P is not Hermitian in the Hilbert space of the Lagrangian displacements. Indeed, the operator adjoint to P is

$$P^A = \begin{pmatrix} -D & -\mathbf{i} \Delta \mathbf{a} \{\} \\ -\mathbf{i} & -D \end{pmatrix}, \quad (11)$$

for it is shown in LM that, for any two six-vectors $\boldsymbol{\zeta}(\mathbf{x}, \mathbf{v})$ and $\boldsymbol{\eta}(\mathbf{x}, \mathbf{v})$, say, we have

$$\langle \boldsymbol{\zeta}, P \boldsymbol{\eta} \rangle_2 = \langle P^A \boldsymbol{\zeta}, \boldsymbol{\eta} \rangle_2. \quad (12)$$

Inasmuch as the characteristic value problem represented by equation (5) is not self-adjoint, it must be solved in association with the solution of the adjoint problem

$$\omega^* \boldsymbol{\eta}^A = P^A \boldsymbol{\eta}^A \quad (13)$$

governing vectors

$$\boldsymbol{\eta}^A = \begin{pmatrix} \Delta \mathbf{v}^A \\ \Delta \mathbf{x}^A \end{pmatrix} \quad (14)$$

adjoint to the characteristic vectors $\boldsymbol{\eta}$. The notation on the right-hand side of equation (14) is intended to emphasize that we let the three-vectors $\Delta \mathbf{v}^A$ and $\Delta \mathbf{x}^A$ have units of a velocity and a length, respectively, so that the unit of an inner product such as $\langle \boldsymbol{\zeta}^A, \boldsymbol{\eta} \rangle_2$, say, is well defined.

As is explained in Section II d of LM, the association of the characteristic value problem with a distinct adjoint problem introduces some complication, in general, into the solution of the characteristic value problem. However, as is further explained in Section III of LM, the situation is greatly simplified in cases in which the structure of the unperturbed system is invariant with respect to the simultaneous application of time reversal and reflection through a plane. For the sake of definiteness we express that invariance by writing

$$f_0(x_1, -x_2, x_3, -v_1, v_2, -v_3) = f_0(x_1, x_2, x_3, v_1, v_2, v_3), \quad (15)$$

where we are letting x_i and v_i ($i = 1, 2, 3$) denote the Cartesian components of \mathbf{x} , and \mathbf{v} , respectively, and we are letting the plane of reflection be the (x_1, x_3) -plane. In this paper, we formulate and study the matrix method for systems having the

symmetry described by equation (15). Examples of such systems include spherically symmetric systems, rotating axisymmetric systems in which the unperturbed distribution function $f_0(\mathbf{x}, \mathbf{v})$ can be expressed in accordance with the theorem of Jeans as a function of the energy integral and the conserved component of the angular momentum of a star, and systems (including axisymmetric systems) with triplanar symmetry in which the gravitational potential $V_0(\mathbf{x})$ is of the Stäckel form and $f_0(\mathbf{x}, \mathbf{v})$ is expressible in terms of three isolating integrals.

The simplification of the characteristic value problem for such systems arises, because it is a consequence of equation (15) that a characteristic vector $\boldsymbol{\eta}(\omega)$ and its adjoint vector $\boldsymbol{\eta}^A(\omega^*)$ satisfy the relation

$$\boldsymbol{\eta}^A(\omega^*) = A\boldsymbol{\eta}(\omega), \quad (16)$$

where A is the matrix

$$A = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}, \quad (17)$$

and the operator C transforms a three-vector function $\mathbf{g}(\mathbf{x}, \mathbf{v}, t)$ (say) in the phase space in the manner

$$\mathbf{g}(\mathbf{x}, \mathbf{v}, t) = \begin{pmatrix} g_1(x_1, x_2, x_3, v_1, v_2, v_3, t) \\ g_2(x_1, x_2, x_3, v_1, v_2, v_3, t) \\ g_3(x_1, x_2, x_3, v_1, v_2, v_3, t) \end{pmatrix} \rightarrow C\mathbf{g}(\mathbf{x}, \mathbf{v}, t) = \begin{pmatrix} g_1^*(x_1, -x_2, x_3, -v_1, v_2, -v_3, -t) \\ -g_2^*(x_1, -x_2, x_3, -v_1, v_2, -v_3, -t) \\ g_3^*(x_1, -x_2, x_3, -v_1, v_2, -v_3, -t) \end{pmatrix}. \quad (18)$$

The simultaneous application of time reversal, reflection through the (x_1, x_3) -plane, and complex conjugation transforms $\Delta\mathbf{x}$ and $\Delta\mathbf{v}$ in the manner

$$\Delta\mathbf{x}(\mathbf{x}, \mathbf{v}, t) \rightarrow C\Delta\mathbf{x}(\mathbf{x}, \mathbf{v}, t) \quad \text{and} \quad \Delta\mathbf{v}(\mathbf{x}, \mathbf{v}, t) \rightarrow -C\Delta\mathbf{v}(\mathbf{x}, \mathbf{v}, t), \quad (19)$$

respectively. The operator C commutes with the operators D and $\Delta\mathbf{a}\{\}$ in the sense that

$$C(D\mathbf{g}) = D(C\mathbf{g}) \quad \text{and} \quad C(\Delta\mathbf{a}\{\mathbf{g}\}) = \Delta\mathbf{a}\{C\mathbf{g}\}, \quad (20)$$

where $\mathbf{g}(\mathbf{x}, \mathbf{v}, t)$ is any three-vector function in the six-dimensional phase space.

3 FORMULATION OF A RAYLEIGH-RITZ MATRIX METHOD

We turn now to the matrix method for the solution of the characteristic value problem. The method is formulated here for normal modes whose frequencies form a discrete spectrum.

3.1 A variational principle

The basis for this matrix method is the following variational principle, which was proved in Section IIIc of LM. Let

$$\boldsymbol{\eta} = \begin{pmatrix} \Delta\mathbf{x} \\ \Delta\mathbf{v} \end{pmatrix} \quad (21)$$

be an arbitrary six-vector in the six-dimensional phase space, and let the value of ω be determined by the relation

$$\langle A\boldsymbol{\eta}, (\omega - P)\boldsymbol{\eta} \rangle_2 = 0. \quad (22)$$

Let $\delta\omega$ be the change of ω that results when $\boldsymbol{\eta}$ is subjected to an arbitrary variation $\delta\boldsymbol{\eta}$ in equation (22). Then, under conditions in which $\langle A\boldsymbol{\eta}, \boldsymbol{\eta} \rangle_2 \neq 0$, a necessary and sufficient condition that $\delta\omega = 0$ is that ω and $\boldsymbol{\eta}$ satisfy the equation

$$\omega\boldsymbol{\eta} = P\boldsymbol{\eta} \quad (23)$$

and, accordingly, that $\boldsymbol{\eta}$ is the characteristic vector belonging to the characteristic frequency ω .

3.2 The matrix method

Let the functions $\boldsymbol{\beta}(\mathbf{x}, \mathbf{v}; k)$ ($k = 1, 2, \dots$) be elements of a complete set of basis vectors in the Hilbert space of the Lagrangian displacements, and let these basis vectors satisfy the orthogonality conditions

$$\langle A\boldsymbol{\beta}(j), \boldsymbol{\beta}(k) \rangle_2 = 0 \quad \text{if} \quad j \neq k. \quad (24)$$

In view of equation (10), we require that the basis vectors also satisfy the divergence condition $\nabla_6 \cdot \boldsymbol{\beta}(k) = 0$. We represent the Lagrangian displacement as a superposition of the basis vectors of the form

$$\boldsymbol{\eta} = \sum_k a(k)\boldsymbol{\beta}(k), \quad (25)$$

where the coefficients $a(k)$ are constants.

In order to apply the variational principle, we identify the right-hand side of equation (25) as a trial function for the Lagrangian displacement in which the coefficients $a(k)$ are the variational parameters. Thus, we substitute from equation (25) for $\boldsymbol{\eta}$ in equation (22), and we make use of the definition of the inner product in equation (9), the definition of the operator A in equations (17) and (18), and the orthogonality relations in equation (24) in order to simplify the resulting equation. We next subject the resulting equation to mutually independent, arbitrary variations $\delta a(k)$ of the coefficients $a(k)$. We require that the variation $\delta\omega$ of the frequency ω vanish in the variational equation that is obtained in this process. As a consequence of the independence and arbitrariness of the variations $\delta a(k)$, the variational equation can be satisfied, in general, only if the frequency ω and the coefficients $a(k)$ satisfy the system of equations

$$\omega \langle A\boldsymbol{\beta}(k), \boldsymbol{\beta}(k) \rangle_2 a(k) = \sum_j \frac{1}{2} [\langle A\boldsymbol{\beta}(j), P\boldsymbol{\beta}(k) \rangle_2 + \langle A\boldsymbol{\beta}(k), P\boldsymbol{\beta}(j) \rangle_2] a(j) \quad (k = 1, 2, \dots). \quad (26)$$

This is the matrix form of the characteristic value problem governing the normal modes of the system in the Lagrangian representation. Equations (26) are a homogeneous system of linear equations in the coefficients $a(k)$. In general, a solution for those coefficients exists if and only if the determinant of the secular matrix of the system vanishes. That condition provides the characteristic equation for the determination of the frequencies ω of the modes.

One might have thought that the matrix representation of the characteristic value problem could be derived directly from equation (5). However, it is not clear how, in such a direct approach, one would know that the orthogonality relations must be formulated in terms of inner products of the basis vectors and their adjoint vectors, that the matrix of the operator P is to be constructed as it is in terms of the basis vectors and their adjoint vectors, or that the matrix of the operator P is to be made symmetric as it is on the right-hand side of equation (26). On the other hand, these conditions are imposed automatically in the present derivation of the matrix equations from the variational principle. Thus the variational principle would appear to be an essential foundation for the matrix method.

We have deliberately retained the inner products $\langle A\boldsymbol{\beta}(k), \boldsymbol{\beta}(k) \rangle_2$ of the basis vectors and their adjoint vectors on the left-hand side of equation (26). Ordinarily, we might expect to normalize the basis vectors so that each of these inner products is equal to unity. However, we have found in practice, as is explained in Section 4 below, that the inner products of some basis vectors with their own adjoint vectors can vanish. In other words, $\langle A\boldsymbol{\beta}(k), \boldsymbol{\beta}(j) \rangle_2$ is a diagonal matrix, in virtue of the orthogonality relations, with some vanishing elements on the principal diagonal. The non-vanishing elements on the principal diagonal may be normalized to unity in the usual way.

Equations (26) are an infinite system of equations. In practice, we approximate the characteristic value problem by expressing the Lagrangian displacement as a superposition of a finite set of basis vectors $\boldsymbol{\beta}(k)$ on the right-hand side of equation (25) and truncating the system of equations (26) accordingly. In the following sections, we consider the solution of the characteristic value problem in such an approximation.

4 A SET OF BASIS VECTORS FOR THE REPRESENTATION OF RADIAL OSCILLATIONS OF A SPHERICAL GALAXY

In the remainder of this paper, we describe and illustrate the matrix method with the aid of an application to radial modes of oscillation of a spherical stellar system. Consistently with the theorem of Jeans, we let the distribution function of the unperturbed system be of the form $f_0(\mathbf{x}, \mathbf{v}) = f_0(E, L^2)$, a function of the energy E and total angular momentum L of a star. Accordingly, the distribution function $f_0(\mathbf{x}, \mathbf{v})$ is an even function of the components of the velocity \mathbf{v} . This section is devoted to a procedure for the construction of the basis vectors required for the representation of radial perturbations in such a sphere.

The essential idea of the procedure is to construct a set of basis vectors which would suffice for the exact solution of the equations of motion governing the normal modes of oscillation of a particular model of a stellar system. The model chosen in the present case is a homogeneous sphere. Although the homogeneous sphere is an unrealistic model of a galaxy, the resulting basis vectors appear to give good results for the frequencies of radial modes even in quite centrally concentrated systems (see Section 6 below).

For a homogeneous sphere of stars in which the unperturbed distribution function $f_0(\mathbf{x}, \mathbf{v}) = f_0(E, L^2)$ is of a suitable form, the radial modes of oscillation have Eulerian perturbations of the gravitational potential which are polynomials in the square of the radial coordinate $r = |\mathbf{x}|$ (see Chapter III, Section 4.1 of Fridman & Polyachenko 1984). The present construction of the basis vectors is accordingly based on solutions of the Lagrangian perturbation equations for the response of a homogeneous sphere to an imposed Eulerian perturbation of the gravitational potential which is a polynomial in r^2 . The solution for the Lagrangian displacement of the response is composed of six-vectors in the phase space which are polynomials in the Cartesian components of the position \mathbf{x} and velocity \mathbf{v} of a star. With a slight generalization for applications to systems with inhomogeneous density distributions, those ‘six-vector polynomials’ are almost the basis vectors that we are seeking. They are divergence free in the phase space, but they are not orthogonal in the sense of equation (24). The set of basis vectors

finally adopted is constructed from the set of vector polynomials with the aid of a Gram-Schmidt procedure for satisfying the orthogonality conditions. Like the vector polynomials, the basis vectors are polynomials in the Cartesian components of \mathbf{x} and \mathbf{v} .

4.1 Perturbations of a homogeneous sphere.

For the purpose of investigating radial perturbations of a homogeneous sphere, it is convenient to reduce the Lagrangian perturbation equations as follows. In the interior of the sphere, the unperturbed gravitational potential is

$$V_0(\mathbf{x}) = \frac{2\pi}{3} G \rho_0 r^2 = \frac{1}{2} n_0^2 r^2, \quad (27)$$

apart from an additive constant, where ρ_0 is the density in the sphere,

$$n_0^2 = \frac{4\pi}{3} G \rho_0, \quad (28)$$

and $r = |\mathbf{x}|$ is the radial coordinate of the point \mathbf{x} in a system of spherical polar coordinates with the origin at the centre of the sphere. Stellar orbits in the unperturbed system are simple harmonic oscillations with frequency n_0 .

In the expression for $\Delta \mathbf{a}\{\Delta \mathbf{x}\}$ in the second of equations (1) (see also eq. [3]), we substitute from equation (27) for $V_0(\mathbf{x})$ and we identify

$$V_1(\mathbf{x}, t) = G m_* \frac{\partial}{\partial \mathbf{x}} \cdot \int_{\Omega} \frac{\Delta \mathbf{x}(\mathbf{x}', \mathbf{v}', t) f_0(\mathbf{x}', \mathbf{v}')}{|\mathbf{x}' - \mathbf{x}|} d\mathbf{x}' d\mathbf{v}' \quad (29)$$

as the Eulerian perturbation of the gravitational potential (see eq.[14] in Vandervoort[1983]). We now eliminate $\Delta \mathbf{v}$ between equations (1), and we bring the resulting equation of motion for $\Delta \mathbf{x}$ to the form

$$\left(\frac{d^2}{dt^2} + n_0^2 \right) \Delta \mathbf{x} = - \frac{\partial V_1}{\partial \mathbf{x}}. \quad (30)$$

In the integration of equation (30) in what follows, we shall make use of the requirement that \mathbf{x} and \mathbf{v} satisfy the unperturbed equations of motion

$$\frac{d\mathbf{x}}{dt} = \mathbf{v} \quad \text{and} \quad \frac{d\mathbf{v}}{dt} = -n_0^2 \mathbf{x}. \quad (31)$$

For a radial oscillation of the system, the Eulerian perturbation of the gravitational potential is of the form $V_1(\mathbf{x}, t) = V_1(r) \exp(-i\omega t)$, where ω is the frequency of the oscillation. When the radial function $V_1(r)$ is imposed in the form of a polynomial in r^2 , the solution of equation (30) is a superposition of the solutions of a set of equations of the form

$$\left(\frac{d^2}{dt^2} + n_0^2 \right) \Delta \mathbf{x} = -\mathbf{x} r^{2k} \exp(-i\omega t) \quad (k = 0, 1, 2, \dots). \quad (32)$$

For the integration of equations (32), we introduce the variables

$$\mathbf{z}_{\sigma} = \mathbf{x} + \sigma i n_0^{-1} \mathbf{v} \quad (\sigma = +1, -1). \quad (33)$$

After writing the right-hand side of equation (32) in terms of the new variables and expanding the resulting expression, we bring equation (32) to the form

$$\left(\frac{d^2}{dt^2} + n_0^2 \right) \Delta \mathbf{x} = - \left(\frac{1}{4} \right)^k \sum_{n=0}^k \sum_{p=0}^{k-n} \sum_{\sigma} 2^{n-1} \binom{k}{n} \binom{k-n}{p} \mathbf{z}_{\sigma} M(\mathbf{z}_{+1}, \mathbf{z}_{-1}; k-n-p, n, p) \exp(-i\omega t), \quad (34)$$

where $\binom{m}{n}$ denotes a binomial coefficient, the sum over σ is over the values $\sigma = +1$ and $\sigma = -1$,

$$M(\mathbf{z}_{+1}, \mathbf{z}_{-1}; m, n, p) = (\mathbf{z}_{+1} \cdot \mathbf{z}_{+1})^m (\mathbf{z}_{+1} \cdot \mathbf{z}_{-1})^n (\mathbf{z}_{-1} \cdot \mathbf{z}_{-1})^p, \quad (35)$$

and $m (= k - n - p)$, n , and p are integers. Therefore, the solution of equation (32) can be written as a superposition of solutions of the equations

$$\left(\frac{d^2}{dt^2} + n_0^2 \right) \Delta \mathbf{x} = -\mathbf{z}_{\sigma} M(\mathbf{z}_{+1}, \mathbf{z}_{-1}; m, n, p) \exp(-i\omega t). \quad (36)$$

In virtue of equations (31), the variables \mathbf{z}_{σ} satisfy the equations of motion

$$\frac{d\mathbf{z}_{\sigma}}{dt} = -\sigma i n_0 \mathbf{z}_{\sigma} \quad (\sigma = +1, -1). \quad (37)$$

With the aid of equations (35) and (37), we readily verify that the particular solution of equation (36) for the response of the system to an imposed ‘Eulerian perturbation of the acceleration’ of the form $-\mathbf{z}_\sigma M(\mathbf{z}_{+1}, \mathbf{z}_{-1}; m, n, p) \exp(-i\omega t)$ is

$$\Delta \mathbf{x}_\sigma(\mathbf{z}_{+1}, \mathbf{z}_{-1}, t; m, n, p) = \frac{\mathbf{z}_\sigma M(\mathbf{z}_{+1}, \mathbf{z}_{-1}; m, n, p) \exp(-i\omega t)}{[\omega + (2m - 2p + \sigma)n_0]^2 - n_0^2}. \quad (38)$$

The Lagrangian perturbation of the velocity is accordingly

$$\Delta \mathbf{v}_\sigma(\mathbf{z}_{+1}, \mathbf{z}_{-1}, t; m, n, p) = \frac{d}{dt} \Delta \mathbf{x}_\sigma(\mathbf{z}_{+1}, \mathbf{z}_{-1}, t; m, n, p) = -i[\omega + (2m - 2p + \sigma)n_0] \Delta \mathbf{x}_\sigma(\mathbf{z}_{+1}, \mathbf{z}_{-1}, t; m, n, p). \quad (39)$$

The Lagrangian displacement described by equations (38) and (39) can now be represented as the six-vector

$$\begin{aligned} \boldsymbol{\eta}_\sigma(\mathbf{z}_{+1}, \mathbf{z}_{-1}, t; m, n, p) &= \begin{pmatrix} \Delta \mathbf{x}_\sigma(\mathbf{z}_{+1}, \mathbf{z}_{-1}, t; m, n, p) \\ \Delta \mathbf{v}_\sigma(\mathbf{z}_{+1}, \mathbf{z}_{-1}, t; m, n, p) \end{pmatrix} \\ &= \frac{1}{[\omega + (2m - 2p + \sigma)n_0]^2 - n_0^2} \begin{pmatrix} \mathbf{z}_\sigma M(\mathbf{z}_{+1}, \mathbf{z}_{-1}; m, n, p) \\ -i[\omega + (2m - 2p + \sigma)n_0] \mathbf{z}_\sigma M(\mathbf{z}_{+1}, \mathbf{z}_{-1}; m, n, p) \end{pmatrix} \exp(-i\omega t). \end{aligned} \quad (40)$$

For the purpose of writing the solutions of equations (32), it is useful to rewrite equation (40) in the form

$$\boldsymbol{\eta}_\sigma(\mathbf{z}_{+1}, \mathbf{z}_{-1}, t; m, n, p) = -\frac{\sigma \boldsymbol{\mu}_\sigma^{+1}(\mathbf{z}_{+1}, \mathbf{z}_{-1}; m, n, p) \exp(-i\omega t)}{2n_0[\omega + (2m - 2p + 2\sigma)n_0]} + \frac{\sigma \boldsymbol{\mu}_\sigma^{-1}(\mathbf{z}_{+1}, \mathbf{z}_{-1}; m, n, p) \exp(-i\omega t)}{2n_0[\omega + (2m - 2p)n_0]}, \quad (41)$$

where the quantities

$$\boldsymbol{\mu}_\sigma^\tau(\mathbf{z}_{+1}, \mathbf{z}_{-1}; m, n, p) = \begin{pmatrix} \mathbf{z}_\sigma M(\mathbf{z}_{+1}, \mathbf{z}_{-1}; m, n, p) \\ \tau \sigma n_0 \mathbf{z}_\sigma M(\mathbf{z}_{+1}, \mathbf{z}_{-1}; m, n, p) \end{pmatrix} \quad (\sigma = +1, -1; \tau = +1, -1) \quad (42)$$

are ‘six-vector monomials’ in the three-vectors \mathbf{z}_σ . Equation (41) represents the required solution of equation (36). Upon comparing equations (34) and (36), we see that the solution of equation (34) is

$$\boldsymbol{\eta}(\mathbf{z}_{+1}, \mathbf{z}_{-1}, t; k) = -\left(\frac{1}{4}\right)^{k+1} \sum_{n=0}^k \sum_{p=0}^{k-n} \sum_{\sigma, \tau} 2^n \binom{k}{n} \binom{k-n}{p} \left(\frac{\tau \sigma \boldsymbol{\mu}_\sigma^\tau(\mathbf{z}_{+1}, \mathbf{z}_{-1}; k-n-p, n, p) \exp(-i\omega t)}{n_0[\omega + (2k-2n-4p+\sigma+\tau\sigma)n_0]} \right), \quad (43)$$

a superposition of the Lagrangian displacements represented by equation (41), where the sum over τ is over the values $\tau = +1$ and $\tau = -1$. In other words, equation (43) represents the Lagrangian displacement that satisfies equation (32) and describes the response of a homogeneous sphere to an ‘Eulerian perturbation of the potential’ equal to $(2k+2)^{-1} r^{2k+2} \exp(-i\omega t)$.

4.2 Vector polynomials and their essential properties

For $k = 0, 1, \dots$, we identify the vector polynomials to be used in the construction of the basis vectors by inspection of equation (43). For a given value of k , we observe that the possible values of the denominators on the right-hand side of equation (43) are $\omega + (2k-2n-4p+\sigma+\tau\sigma)n_0 = \omega + (2k+2)n_0, \omega + 2kn_0, \omega + (2k-2)n_0, \dots, \omega - (2k+2)n_0$, apart from a common factor n_0 . The number of distinct values of such denominators for a given value of k is equal to $2k+3$. Now we have obtained equation (43) as a solution of equation (30) (see also equation [32]) for an arbitrarily imposed Eulerian perturbation of the gravitational acceleration. In particular, the frequency ω has an arbitrary value. On the other hand, the basis vectors and their ingredients should be independent of the value of ω . Therefore, we suppress the factor $\exp(-i\omega t)$ on the right-hand side of equation (43), and we identify the sum of the numerators there for each distinct value of the denominator $\omega + (2k-2n-4p+\sigma+\tau\sigma)n_0$ as a vector polynomial. There are thus $2k+3$ vector polynomials for a given power k of r^2 on the right-hand side of equation (32). This procedure determines the vector polynomials apart from arbitrary normalization factors.

In the case that $k = 0$, for example, equation (43) can be written explicitly as

$$\begin{aligned} \boldsymbol{\eta}(\mathbf{z}_{+1}, \mathbf{z}_{-1}, t; 0) &= -\frac{1}{4n_0} \left(\frac{\boldsymbol{\mu}_{+1}^{+1}(\mathbf{z}_{+1}, \mathbf{z}_{-1}; 0, 0, 0)}{\omega + 2n_0} - \frac{\boldsymbol{\mu}_{+1}^{-1}(\mathbf{z}_{+1}, \mathbf{z}_{-1}; 0, 0, 0)}{\omega} - \frac{\boldsymbol{\mu}_{-1}^{+1}(\mathbf{z}_{+1}, \mathbf{z}_{-1}; 0, 0, 0)}{\omega - 2n_0} + \frac{\boldsymbol{\mu}_{-1}^{-1}(\mathbf{z}_{+1}, \mathbf{z}_{-1}; 0, 0, 0)}{\omega} \right), \end{aligned} \quad (44)$$

where we have suppressed the factor $\exp(-i\omega t)$ on the right-hand side. An inspection of the right-hand side of equation (44) as described in the preceding paragraph yields the three vector polynomials

$$\begin{aligned} \boldsymbol{\pi}(\mathbf{z}_{+1}, \mathbf{z}_{-1}; 0, 1) &= \boldsymbol{\mu}_{+1}^{+1}(\mathbf{z}_{+1}, \mathbf{z}_{-1}; 0, 0, 0), \quad \boldsymbol{\pi}(\mathbf{z}_{+1}, \mathbf{z}_{-1}; 0, 2) = \boldsymbol{\mu}_{-1}^{+1}(\mathbf{z}_{+1}, \mathbf{z}_{-1}; 0, 0, 0), \quad \text{and} \\ \boldsymbol{\pi}(\mathbf{z}_{+1}, \mathbf{z}_{-1}; 0, 3) &= \boldsymbol{\mu}_{-1}^{-1}(\mathbf{z}_{+1}, \mathbf{z}_{-1}; 0, 0, 0) - \boldsymbol{\mu}_{+1}^{-1}(\mathbf{z}_{+1}, \mathbf{z}_{-1}; 0, 0, 0). \end{aligned} \quad (45)$$

In the notation $\boldsymbol{\pi}(\mathbf{z}_{+1}, \mathbf{z}_{-1}; k, h)$ for a vector polynomial, we adopt the convention that the integer k identifies the *group* of vector polynomials to which $\boldsymbol{\pi}(\mathbf{z}_{+1}, \mathbf{z}_{-1}; k, h)$ belongs, i.e., the power k of r^2 on the right-hand side of equation (32), and the value of the integer h ($h = 1, 2, \dots, 2k+3$) distinguishes $\boldsymbol{\pi}(\mathbf{z}_{+1}, \mathbf{z}_{-1}; k, h)$ from the other members of that group. In equations (A1)-(A3) in Appendix A, we list the expressions for the additional vector polynomials through the group $k = 3$.

We choose the normalization factors in the vector polynomials listed in equations (45) and (A1)-(A3) and we assign the order in which to list the vector polynomials in each group so that most of the vector polynomials are conveniently arranged in complex-conjugate pairs (see eq. [51] below).

The vector polynomials listed in equations (45) and (A1)-(A3) would be ingredients for an exact construction of normal modes in the homogeneous spheres described by Fridman and Polyachenko (1984, see Chapter III, Section 4.1). In order to make use of the six-vector monomials $\boldsymbol{\mu}_\sigma^\tau(m, n, p)$ and the six-vector polynomials $\boldsymbol{\pi}(k, h)$ in the representation of perturbations in inhomogeneous spheres, we must generalize the definition of n_0 . A suitable generalization is provided by the equation

$$n_0^2 = -\frac{W}{I}, \quad (46)$$

where

$$W = -\int_V \rho_0(r) r \frac{\partial V_0(r)}{\partial r} dx \quad \text{and} \quad I = \int_V \rho_0(r) r^2 dx \quad (47)$$

denote the gravitational potential energy and the moment of inertia, respectively, of the unperturbed configuration, $\rho_0(r)$ and $V_0(r)$ are the density and gravitational potential, respectively, of the unperturbed configuration, and the integrations extend over the volume V of the unperturbed configuration. For homogeneous spheres, equations (28) and (46) are equivalent definitions of n_0 . In general, however, the variables \mathbf{z}_σ , the scalar monomials $M(\mathbf{z}_{+1}, \mathbf{z}_{-1}; m, n, p)$, the vector monomials $\boldsymbol{\mu}_\sigma^\tau(\mathbf{z}_{+1}, \mathbf{z}_{-1}; m, n, p)$, and the vector polynomials $\boldsymbol{\pi}(\mathbf{z}_{+1}, \mathbf{z}_{-1}; k, h)$ ($k = 0, \dots, 3$) are defined by equations (33), (35), (42), and (45) and (A1)-(A3), respectively, with n_0 defined by equation (46).

Making use of equations (33), (35), and (42) in explicit calculations applied to the expressions for the vector polynomials listed in equations (45) and (A1)-(A3), we find that

$$\nabla_6 \cdot \boldsymbol{\pi}(k, h) = 0. \quad (48)$$

Thus, the construction of the basis vectors from the vector polynomials will ensure that the representation of the Lagrangian displacement in equation (25) will automatically satisfy equation (10).

In the calculations that follow, we shall need the complex conjugates of the vector polynomials and the vectors adjoint to the vector polynomials. These are to be constructed in terms of the vector monomials, their complex conjugates and their adjoints. Referring to equations (33), (35), and (42), respectively, we find that

$$\mathbf{z}_\sigma^* = \mathbf{z}_{-\sigma}, \quad M^*(\mathbf{z}_{+1}, \mathbf{z}_{-1}; m, n, p) = M(\mathbf{z}_{+1}, \mathbf{z}_{-1}; p, n, m), \quad \text{and} \quad [\boldsymbol{\mu}_\sigma^\tau(\mathbf{z}_{+1}, \mathbf{z}_{-1}; m, n, p)]^* = \boldsymbol{\mu}_{-\sigma}^\tau(\mathbf{z}_{+1}, \mathbf{z}_{-1}; p, n, m), \quad (49)$$

where the asterisk, written as a superscript, labels the complex conjugate of a quantity. With the aid of equations (16)-(18), (33), and (49), we can also verify that $C[\mathbf{z}_\sigma M(\mathbf{z}_{+1}, \mathbf{z}_{-1}; m, n, p)] = \mathbf{z}_\sigma M(\mathbf{z}_{+1}, \mathbf{z}_{-1}; m, n, p)$ and, accordingly, that

$$A\boldsymbol{\mu}_\sigma^\tau(\mathbf{z}_{+1}, \mathbf{z}_{-1}; m, n, p) = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} \begin{pmatrix} \mathbf{z}_\sigma M(\mathbf{z}_{+1}, \mathbf{z}_{-1}; m, n, p) \\ \tau \sigma i n_0 \mathbf{z}_\sigma M(\mathbf{z}_{+1}, \mathbf{z}_{-1}; m, n, p) \end{pmatrix} = \begin{pmatrix} -\tau \sigma i n_0 \mathbf{z}_\sigma M(\mathbf{z}_{+1}, \mathbf{z}_{-1}; m, n, p) \\ \mathbf{z}_\sigma M(\mathbf{z}_{+1}, \mathbf{z}_{-1}; m, n, p) \end{pmatrix}. \quad (50)$$

The third of equations (49) implies that the vector polynomials listed in equations (45) and (A1)-(A3) satisfy the complex-conjugate relations

$$\boldsymbol{\pi}(k, h) = \boldsymbol{\pi}^*(k, h-1) \quad (h = 2, 4, \dots, 2k+2) \quad \text{and} \quad \boldsymbol{\pi}(k, 2k+3) = -\boldsymbol{\pi}^*(k, 2k+3). \quad (51)$$

In other words, the group of vector polynomials for a given value of k contains $k+1$ complex-conjugate pairs of members and one imaginary member.

For the explicit construction of the basis vectors and for the associated numerical calculations, we relabel the vector polynomials in terms of a single integer. The prescription for doing so depends on the number of groups of vector polynomials to be used in the construction of the basis vectors. Thus, in the following subsection, we shall construct the basis vectors from the groups of vector polynomials $\boldsymbol{\pi}(k, h)$ for which $k = 0, 1, \dots, k_{\max}$, say.

We first relabel the vector polynomials which form complex-conjugate pairs by writing (see eqs. [51])

$$\boldsymbol{\pi}(\mathbf{z}_{+1}, \mathbf{z}_{-1}; k, h) = \boldsymbol{\pi}(\mathbf{z}_{+1}, \mathbf{z}_{-1}; q), \quad \text{where} \quad q = k(k+1) + h \quad (k = 0, 1, \dots, k_{\max}; h = 1, 2, \dots, 2k+2). \quad (52)$$

The value of the new integer q to be associated with a given pair of values (k, h) is determined here as follows. We recall that the group of vector polynomials $\boldsymbol{\pi}(k', h')$, for a given value of k' ($k' < k$), contains $2k' + 2$ complex members. Summing that number from $k' = 0$ to $k' = k-1$, we count a total of $k(k+1)$ complex vector polynomials with $0 \leq k' \leq k-1$. We then let $q = k(k+1) + h$, inasmuch as the polynomial $\boldsymbol{\pi}(k, h)$ is number h in the listing of the polynomials belonging to the group k . The integer q takes the values $q = 1, 2, \dots, Q$ in equations (52), where $Q = (k_{\max} + 1)(k_{\max} + 2)$ is the total number of complex vector polynomials in the groups such that $0 \leq k \leq k_{\max}$. We next add the purely imaginary vector polynomials to the enumeration in equation (52) by writing

$$\boldsymbol{\pi}(\mathbf{z}_{+1}, \mathbf{z}_{-1}; k, 2k+3) = \boldsymbol{\pi}(\mathbf{z}_{+1}, \mathbf{z}_{-1}; q), \quad \text{where} \quad q = Q + k + 1 \quad (k = 0, 1, \dots, k_{\max}). \quad (53)$$

Here, the expression for q is determined by the circumstances that we are continuing the enumeration of vector polynomials begun in equation (52) and each group of vector polynomials contributes one purely imaginary member to the enumeration (see eq.[51]). The values of the integer q in equation (53) are $q = Q+1, Q+2, \dots, N$, where $N = Q + k_{\max} + 1 = (k_{\max} + 1)(k_{\max} + 3)$ is the total number of vector polynomials, complex and purely imaginary, in the groups such that $0 \leq k \leq k_{\max}$.

Rewriting equations (51) in accordance with equations (52) and (53), we bring the complex-conjugate relations satisfied by the vector polynomials to the form

$$\pi(q) = \pi^*(q-1) \quad (q = 2, 4, \dots, Q) \quad \text{and} \quad \pi(q) = -\pi^*(q) \quad (q = Q+1, Q+2, \dots, N). \quad (54)$$

According to equations (45) and (A1)-A3), the vector polynomials are superpositions of vector monomials of the form

$$\pi(z_{+1}, z_{-1}; q) = \sum_{j=1}^J \langle q|j \rangle \mu_{\sigma}^{\tau}(z_{+1}, z_{-1}; m, n, p), \quad (55)$$

where we label the vector polynomials $\pi(z_{+1}, z_{-1}; q)$ in terms of the single integer q in accordance with equations (52) and (53). On the right-hand sides of equations (55), we determine the constant coefficients $\langle q|j \rangle$, we determine the integers which distinguish the vector monomials $\mu_{\sigma}^{\tau}(z_{+1}, z_{-1}; m, n, p)$ as functions $\sigma = \sigma(q; j)$, $\tau = \tau(q; j)$, $m = m(q; j)$, $n = n(q; j)$, and $p = p(q; j)$, and we determine the number of terms $J = J(q)$ in each sum by inspection of equations (45) and (A1)-A3). The coefficients $\langle q|j \rangle$ are seen there to be real. It follows that

$$\pi^*(z_{+1}, z_{-1}; q) = \sum_{j=1}^J \langle q|j \rangle [\mu_{\sigma}^{\tau}(z_{+1}, z_{-1}; m, n, p)]^* \quad \text{and} \quad A\pi(z_{+1}, z_{-1}; q) = \sum_{j=1}^J \langle q|j \rangle A\mu_{\sigma}^{\tau}(z_{+1}, z_{-1}; m, n, p). \quad (56)$$

4.3 Construction of the basis vectors

We now write the basis vectors in the form

$$\beta(1) = N(1)\pi(1) \quad \text{and} \quad \beta(q) = N(q) \left[\sum_{r=1}^{q-1} c(q, r)\beta(r) + \pi(q) \right] \quad (q = 2, 3, \dots, N), \quad (57)$$

where the constants $N(q)$ and $c(q, r)$ are to be determined with the aid of a Gram-Schmidt process. In that process, we require that the basis vectors be orthogonal in the sense of equation (24). Thus, for example, when we form the inner products of $A\beta(s)$ ($s \leq q$) with both sides of the second of equations (57) and apply the orthogonality conditions, we obtain

$$N(q) \left[\langle A\beta(s), \beta(s) \rangle_2 c(q, s) + \langle A\beta(s), \pi(q) \rangle_2 \right] = 0 \quad \text{and} \quad \langle A\beta(q), \beta(q) \rangle_2 = N(q) \langle A\beta(q), \pi(q) \rangle_2 \quad (58)$$

$(q = 2, 3, \dots, N; s = 1, 2, \dots, q-1).$

For the reduction of equations (58) to forms suitable for the systematic evaluation of $N(q)$ and $c(q, s)$, we need the equations

$$A\beta(1) = N^*(1)A\pi(1) \quad \text{and} \quad A\beta(q) = N^*(q) \left[\sum_{r=1}^{q-1} c^*(q, r)A\beta(r) + A\pi(q) \right] \quad (q = 2, 3, \dots, N) \quad (59)$$

for the vectors adjoint to the basis vectors $\beta(q)$. We obtain equations (59) by transforming equations (57) with the aid of equations (17) and (18). Equations (57) describe a representation of the basis vectors in which the constants $N(q)$ must not vanish. Consequently the second of equations (59) allows us to reduce the first of equations (58) (except when $q = 2$ and $s = 1$) to

$$\langle A\beta(s), \beta(s) \rangle_2 c(q, s) = -N(s) \left[\sum_{r=1}^{s-1} c(s, r) \langle A\beta(r), \pi(q) \rangle_2 + \langle A\pi(s), \pi(q) \rangle_2 \right] \quad (q = 3, 4, \dots, N; s = 2, 3, \dots, q-1). \quad (60)$$

The first of equations (58), again with the common factor $N(q)$ suppressed, enables us to rewrite equation (60) as

$$\langle A\beta(s), \beta(s) \rangle_2 c(q, s) = N(s) \left[\sum_{r=1}^{s-1} \langle A\beta(r), \beta(r) \rangle_2 c(s, r) c(q, r) - \langle A\pi(s), \pi(q) \rangle_2 \right] \quad (q = 3, 4, \dots, N; s = 2, 3, \dots, q-1). \quad (61)$$

Likewise, with the aid of the second of equations (59) and then the first of equations (58), we reduce the second of equations (58) to the form

$$\langle A\beta(q), \beta(q) \rangle_2 = N^2(q) \left[\langle A\pi(q), \pi(q) \rangle_2 - \sum_{r=1}^{q-1} \langle A\beta(r), \beta(r) \rangle_2 c^2(q, r) \right] \quad (q = 2, 3, \dots, N). \quad (62)$$

Finally, making use of the first of equations (59) in order to rewrite the first of equations (58) in the case that $s = 1$ and then the second of equations (58) in the case that $q = 1$, we obtain

$$\langle A\beta(1), \beta(1) \rangle_2 c(q, 1) = -N(1) \langle A\pi(1), \pi(q) \rangle_2 \quad (q = 2, 3, \dots, N) \quad \text{and} \quad \langle A\beta(1), \beta(1) \rangle_2 = N^2(1) \langle A\pi(1), \pi(1) \rangle_2 \quad (63)$$

as special cases of equations (61) and (62), respectively. The inner products $\langle A\pi(q'), \pi(q) \rangle_2$ in equations (61)-(63) are given quantities determined by the unperturbed structure of the system (see eq.[9]). Expressions required for the evaluation of $\langle A\pi(q'), \pi(q) \rangle_2$ are derived in Appendix B, and the symmetries and other important properties of $\langle A\pi(q'), \pi(q) \rangle_2$ are described there.

Equations (61)-(63) are the basic equations for the determination of the constants $N(q)$ and $c(q, s)$ in the representation of the basis vectors introduced in equations (57). Before we can solve those equations, we must normalize the basis vectors. In what follows, we shall find that the appropriate normalization is

$$\begin{aligned} \langle A\beta(q), \beta(q) \rangle_2 &= 1 & (q = 1, 2, \dots, Q) \\ &= 0 & (q = Q + 1, Q + 2, \dots, N). \end{aligned} \quad (64)$$

The unusual result that some of the inner products $\langle A\beta(q), \beta(q) \rangle_2$ must vanish complicates the matrix method somewhat, but it seems not to create fundamental difficulties in the application of the matrix method.

With the normalization of the basis vectors given, we can solve equations (61)-(63) for the quantities $N(q)$ and $c(q, s)$ in the following sequence. We first evaluate $N(1)$ and $c(q, 1)$ ($q = 2, 3, \dots, N$) in accordance with equations (63), and we next evaluate $N(2)$ in accordance with the first of equations (62). We subsequently solve equations (61) and (62) in the sequence $q = 3, 4, \dots, N$. For each value of q , in turn, we evaluate the constants $c(q, s)$ in the sequence $s = 2, 3, \dots, q - 1$ with the aid of equations (61), and we finally evaluate $N(q)$ in accordance with equation (62). At each step in this process, the values of $c(s, r)$ and $N(s)$ required for the evaluation of the right-hand side of equation (61) or the coefficient of $N^2(q)$ on the right-hand side of equation (62) are known as results of earlier steps in the process.

In Sections 4.3.1 and 4.3.2 which follow, we present separate discussions of the solutions of equations (61)-(63) for the quantities $N(q)$ and $c(q, s)$ involved in the representations of the basis vectors satisfying the different normalization conditions in equations (64). The solutions for $N(q)$ and $c(q, s)$ satisfy certain identities which are derived in Appendix C. Some of the results described in Sections 4.3.1 and 4.3.2 and in Appendix C follow, because the solutions of equations (61)-(63) have the property that the quantities $N^2(q)$, $c(q, s)/N(s)$, $c(s, r)c(q, r)$, and $c^2(q, r)$ are all imaginary. For a proof of this result, we note that the inner products $\langle A\pi(s), \pi(q) \rangle_2$ are imaginary quantities, as is shown in Appendix B, and we observe that we have normalized the basis vectors so that the inner products $\langle A\beta(s), \beta(s) \rangle_2$ are real quantities. Noting that equations (63) establish the results claimed here for $N(1)$ and $c(q, 1)$, we can prove the general result by induction with the aid of equations (61) and (62).

4.3.1 Complex-conjugate pairs of basis vectors

When $0 < q \leq Q$ and $0 < q' \leq Q$, the inner products $\langle A\pi(q'), \pi(q) \rangle_2$ do not vanish, in general. With the normalization given in equations (64), we find that we can satisfy equations (61)-(63) consistently when $0 < q \leq Q$. In particular, we can solve equations (63) in the manner

$$N^2(1) = [\langle A\pi(1), \pi(1) \rangle_2]^{-1} \quad \text{and} \quad c(q, 1) = -N(1) \langle A\pi(1), \pi(q) \rangle_2 \quad (q = 2, 3, \dots, Q). \quad (65)$$

We can also bring the subsets of equations (61) and (62) with $q \leq Q$ to the forms

$$c(q, s) = N(s) \left[\sum_{r=1}^{s-1} c(s, r)c(q, r) - \langle A\pi(s), \pi(q) \rangle_2 \right] \quad (q = 3, 4, \dots, Q; s = 2, 3, \dots, q - 1) \quad (66)$$

and

$$N^2(q) = \left[\langle A\pi(q), \pi(q) \rangle_2 - \sum_{r=1}^{q-1} c^2(q, r) \right]^{-1} \quad (q = 2, 3, \dots, Q), \quad (67)$$

respectively. Equations (65)-(67) enable us to evaluate the constants $c(q, s)$ and $N(q)$, in the case that $q \leq Q$, in the sequence prescribed above in the paragraph following the normalization of the basis vectors in equations (64).

An important consequence of equations (65)-(67) is that the basis vectors $\beta(q)$ ($q \leq Q$) form complex-conjugate pairs in the arrangement

$$\beta(q) = \beta^*(q - 1) \quad (q = 2, 4, \dots, Q). \quad (68)$$

The expressions for the basis vectors given by equations (57) satisfy equations (68) because the vector polynomials $\pi(q)$ satisfy equations (54) and the quantities determined by equations (65)-(67) satisfy the identities

$$\begin{aligned} N(q) &= N^*(q-1) \quad \text{and} \quad c(q, q-1) = 0 \quad (q = 2, 4, \dots, Q) \quad \text{and} \\ c(q, s) &= c^*(q-1, s-1) \quad \text{and} \quad c(q, s-1) = c^*(q-1, s) \quad (q = 4, 6, \dots, Q; s = 2, 4, \dots, q-2). \end{aligned} \quad (69)$$

These identities are verified in Appendix C.

4.3.2 Null basis vectors

It remains to solve equations (61)-(63) in the case that $Q < q \leq N$. As an essential part of that solution, we must show that the resulting basis vectors are null vectors as we have claimed in equation (64).

When $s \leq Q < q \leq N$, the normalization of the basis vectors $\langle A\beta(s), \beta(s) \rangle_2 = 1$ ($s = 1, 2, \dots, Q$) reduces the first of equations (63) and equations (61) to

$$c(q, 1) = -N(1) \langle A\pi(1), \pi(q) \rangle_2 \quad (q = Q+1, Q+2, \dots, N) \quad (70)$$

and

$$c(q, s) = N(s) \left[\sum_{r=1}^{s-1} c(s, r) c(q, r) - \langle A\pi(s), \pi(q) \rangle_2 \right] \quad (q = Q+1, Q+2, \dots, N; s = 2, 3, \dots, Q), \quad (71)$$

respectively. Starting with the known values of $c(q, r)$ and $N(q)$, evaluated, as described Section 4.3.1, with the aid of equations (65)-(67), we can make use of equations (70) and (71) in order to evaluate the additional quantities $c(q, s)$ in a sequence such that, at each step in the process, the values of quantities required for the evaluation of the right-hand side of equation (71) are known as results of preceding steps.

We must finally consider the solution of equations (61) and (62) in the case that $Q < s < q \leq N$. In this case, the inner products $\langle A\pi(s), \pi(q) \rangle_2$ and $\langle A\pi(q), \pi(q) \rangle_2$ vanish according to equations (B7). The further reduction of equations (61) and (62) requires additional identities, especially equations (C11), which are also derived in Appendix C. In virtue of the normalization $\langle A\beta(q), \beta(q) \rangle_2 = 1$ ($q = 1, 2, \dots, Q$) and equations (C11), the sums of terms from $r = 1$ to $r = Q$ vanish on the right-hand sides of equations (61) and (62). Thus, the subset of equations (61) and (62) which we must finally consider reduces to

$$\langle A\beta(s), \beta(s) \rangle_2 c(q, s) = N(s) \sum_{r=Q+1}^{s-1} \langle A\beta(r), \beta(r) \rangle_2 c(s, r) c(q, r) \quad (q = Q+2, Q+3, \dots, N; s = Q+1, Q+2, \dots, q-1), \quad (72)$$

$$\langle A\beta(Q+1), \beta(Q+1) \rangle_2 = 0, \quad (73)$$

and

$$\langle A\beta(q), \beta(q) \rangle_2 = -N^2(q) \sum_{r=Q+1}^{q-1} \langle A\beta(r), \beta(r) \rangle_2 c^2(q, r) \quad (q = Q+2, Q+3, \dots, N). \quad (74)$$

According to equation (74), if $\langle A\beta(r), \beta(r) \rangle_2 = 0$ ($r = Q+1, \dots, q-1$), then $\langle A\beta(q), \beta(q) \rangle_2 = 0$. Equation (73) shows that the required condition is satisfied for $q = Q+1$. Therefore, by induction, we have $\langle A\beta(q), \beta(q) \rangle_2 = 0$ ($q = Q+1, Q+2, \dots, N$) as is claimed in equation (64). Consequently, equations (72) and (74) leave the values of the remaining quantities $c(q, s)$ and $N(q)$ indeterminate. Without loss of generality, we assign those values in the manner

$$c(q, s) = 0 \quad (q = Q+2, Q+3, \dots, N; s = Q+1, Q+2, \dots, q-1) \quad (75)$$

and

$$N(q) = 1 + i \quad (q = Q+1, Q+2, \dots, N). \quad (76)$$

The values of $N(q)$ adopted in equations (76) preserve the property that the quantities $N^2(q)$ are imaginary.

4.4 The completeness of the basis vectors

The question arises as to whether or not the set of basis vectors constructed here is complete. We have not established that basis vectors constructed in this way form a complete set in general. We can only say that the adopted set of basis vectors is ‘sufficiently complete’ for the exact representation of radial modes in certain homogeneous spheres. Accordingly, one might hope that they would suffice for the representation of corresponding modes in inhomogeneous spheres. It appears in Section 6 that they do indeed suffice for the representation of the lowest radial modes in rather centrally concentrated spheres.

The validity of the present formulation of the matrix method as a method for the approximate solution of the characteristic value problem governing modes does not require that the adopted basis vectors form a complete set. For the matrix method is a particular realization of a variational method in which the basis vectors are the ingredients of a trial function. The essential condition to be satisfied by the basis vectors is that they provide a sufficiently accurate representation of the Lagrangian displacements of the modes to be investigated. This is essentially the point of view adopted previously by Saha (1991).

4.5 Construction of the matrix equations

The construction of a suitable set of basis vectors reduces a given application of the matrix method to the evaluation of the matrix elements in a truncated set of equations (26) and then to the solution of the matrix equations. We describe the construction of the required matrix elements in Appendix D.

5 SOLUTION OF THE CHARACTERISTIC VALUE PROBLEM IN THE MATRIX REPRESENTATION

In the remainder of this paper, we shall restrict ourselves to the consideration of radial oscillations in systems in which the distribution function is a function of the energy alone and the velocity distribution is isotropic. The quadratures required for the evaluation of inner products and matrix elements are described in Appendix E. In what follows, we describe the reduction and solution of the matrix equations.

5.1 Reduction of the matrix equations

With the truncation of the set of basis vectors to a finite set containing N members, and in virtue of the orthogonality and normalization of the basis vectors specified in equations (24) and (64), respectively, the matrix formulation of the characteristic value problem represented in equation (26) now reduces to

$$\omega a(k) = \sum_{j=1}^N R^{(N)}(k, j) a(j) \quad (k = 1, 2, \dots, Q) \quad \text{and} \quad 0 = \sum_{j=1}^N R^{(N)}(k, j) a(j) \quad (k = Q + 1, Q + 2, \dots, N), \quad (77)$$

where

$$R^{(N)}(k, j) = \frac{1}{2} [\langle A\beta(j), P\beta(k) \rangle_2 + \langle A\beta(k), P\beta(j) \rangle_2]. \quad (78)$$

We reduce the system of equations (77) to a standard form by making use of those members of the system in which $k = Q + 1, Q + 2, \dots, N$ in order to eliminate the amplitudes $a(k)$ ($k = Q + 1, Q + 2, \dots, N$) from the remaining equations in the system. We accomplish this with the aid of a sequence of transformations in each of which we eliminate one amplitude. After $N - K$ steps in that process, where K ($Q + 1 \leq K \leq N$) is an integer, we will have reduced equations (77) to a system of the form

$$\omega a(k) = \sum_{j=1}^K R^{(K)}(k, j) a(j) \quad (k = 1, 2, \dots, Q) \quad \text{and} \quad 0 = \sum_{j=1}^K R^{(K)}(k, j) a(j) \quad (k = Q + 1, Q + 2, \dots, K), \quad (79)$$

say, where the matrix $R^{(K)}$ is derived from the matrix $R^{(N)}$ in a sequence of transformations of the form of equation (82) below. Equation (82) is the transformation with which we eliminate $a(K)$. Thus, from the last of equations (79), we derive

$$a(K) = -\frac{1}{R^{(K)}(K, K)} \sum_{j=1}^{K-1} R^{(K)}(K, j) a(j). \quad (80)$$

Upon substituting from this solution for $a(K)$ in equations (79) ($k = 1, 2, \dots, K - 1$), we obtain the system

$$\omega a(k) = \sum_{j=1}^{K-1} R^{(K-1)}(k, j) a(j) \quad (k = 1, 2, \dots, Q) \quad \text{and} \quad 0 = \sum_{j=1}^{K-1} R^{(K-1)}(k, j) a(j) \quad (k = Q + 1, Q + 2, \dots, K - 1), \quad (81)$$

where

$$R^{(K-1)}(k, j) = R^{(K)}(k, j) - \frac{R^{(K)}(k, K) R^{(K)}(K, j)}{R^{(K)}(K, K)}. \quad (82)$$

At the beginning of this sequence of transformations we have $K = N$, and equations (79) coincide with equations (77). At the end of the sequence, we have $K = Q + 1$, and equations (81) reduce to

$$\omega a(k) = \sum_{j=1}^Q R^{(Q)}(k, j) a(j) \quad (k = 1, 2, \dots, Q). \quad (83)$$

We are thus left with the characteristic value problem for the $Q \times Q$ matrix $R^{(Q)}(k, j)$, where $Q = (k_{\max} + 1)(k_{\max} + 2)$.

By definition (see eq. [78]) the matrix $R^{(N)}(k, j)$ has the symmetry $R^{(N)}(j, k) = R^{(N)}(k, j)$. It follows that the matrices defined by equations (82) have the symmetry $R^{(K)}(j, k) = R^{(K)}(k, j)$. In particular, $R^{(Q)}(j, k) = R^{(Q)}(k, j)$. Moreover, in the numerical investigation of the solutions of equation (83), we find that the matrix $R^{(Q)}(k, j)$ has the following additional properties to the accuracy in which that matrix is evaluated.

- (i) For even values of $k - j$, the elements $R^{(Q)}(k, j)$ are real.
- (ii) For odd values of $k - j$, the elements $R^{(Q)}(k, j)$ are imaginary.

Thus the complex matrix $R^{(Q)}(k, j)$ can be expressed in terms of a real matrix $S^{(Q)}(k, j)$, say, in the manner

$$R^{(Q)}(k, j) = i^{(k-j)} S^{(Q)}(k, j). \quad (84)$$

It follows from this definition of $S^{(Q)}(k, j)$ and the symmetry of $R^{(Q)}(k, j)$ that

$$S^{(Q)}(k, j) = (-i)^{(k-j)} R^{(Q)}(k, j) \quad \text{and} \quad S^{(Q)}(j, k) = (-1)^{(k-j)} S^{(Q)}(k, j). \quad (85)$$

Letting

$$a(k) = i^k \alpha(k) \quad (k = 1, 2, \dots, Q), \quad (86)$$

where the quantities $\alpha(k)$ are constants, we can now transform the system of equations (83) to the system

$$\omega \alpha(k) = \sum_{j=1}^Q S^{(Q)}(k, j) \alpha(j) \quad (k = 1, 2, \dots, Q), \quad (87)$$

and we are left with a characteristic value problem for the real matrix $S^{(Q)}(k, j)$.

The numerical calculations required for the evaluation of the matrix $S^{(Q)}(k, j)$ and the solution of equations (87) have been performed with the aid of programs written in FORTRAN 77 and run in single-precision and complex arithmetic on Macintosh II and Macintosh IIfx computers. The program for the solution of equations (87) incorporates subroutines from the NAPAK library. Numerical quadratures required in the evaluation of inner products and matrix elements (see Appendix E) were performed with the aid of the fourth order Runge-Kutta integrator described by Press et al. (1986).

These calculations have been performed in the four approximations defined by setting k_{\max} equal to 0, 1, 2, and 3. The corresponding values of $Q = (k_{\max} + 1)(k_{\max} + 2)$ are 2, 6, 12, and 20, respectively. In other words, we have solved equations (87) for 2 by 2, 6 by 6, 12 by 12, and 20 by 20 matrices $S^{(Q)}(k, j)$, respectively. For an assigned value of k_{\max} , the number of basis vectors included in the representation of the Lagrangian displacement is $N = (k_{\max} + 1)(k_{\max} + 3)$. Thus, we include 3, 8, 15, and 24 basis vectors in the four approximations considered here. The numerical calculations required for the evaluation of $S^{(Q)}(k, j)$ and the solution of equations (87) in all four approximations require about four hours for one unperturbed model.

5.2 Characteristic vectors

Let $\omega(n)$ ($n = 1, 2, \dots, Q$) denote the solutions of equations (87) for the characteristic frequencies for a given value of Q , and let $\alpha(j, n)$ ($j = 1, 2, \dots, Q; n = 1, 2, \dots, Q$) denote the corresponding solutions for the constants $\alpha(j)$. The solution for the Lagrangian displacement which belongs to the characteristic frequency $\omega(n)$ is then

$$\boldsymbol{\eta}(n) = \sum_{j=1}^N a(j, n) \boldsymbol{\beta}(j) \quad (88)$$

(cf. eq. [25]). Here,

$$a(j, n) = i^j \alpha(j, n) \quad (j = 1, 2, \dots, Q) \quad (89)$$

(cf. eq. [86]), and the remaining values of the constants $a(j, n)$ ($j = Q + 1, Q + 2, \dots, N$) are determined by working backward (i.e., in the sequence $K = Q + 1, Q + 2, \dots, N$) through equations (80)-(82) with the aid of the values of the first Q constants $a(j, n)$.

Equation (88) represents a characteristic solution for the Lagrangian displacement in the approximation that the right-hand side of equation (25) is truncated to a superposition of N basis vectors. These characteristic solutions satisfy orthogonality relations of the form

$$\langle A\boldsymbol{\eta}(m), \boldsymbol{\eta}(n) \rangle_2 = \delta_{mn}, \quad (90)$$

where δ_{mn} is the Kronecker delta. In order to verify equations (90), we first substitute from equations (88) for the Lagrangian displacements in the inner product on the left-hand side of equation (90) and bring the resulting expression to the form

$$\langle A\boldsymbol{\eta}(m), \boldsymbol{\eta}(n) \rangle_2 = \sum_{j=1}^Q a(j, m) a(j, n) \quad (91)$$

with the aid of equations (24) and (64). Now the requirement that the quantities $\omega(n)$ and $\alpha(j, n)$ ($j = 1, 2, \dots, Q; n = 1, 2, \dots, Q$) be a characteristic solution of equations (87) implies that the quantities $\omega(n)$ and $a(j, n)$ ($j = 1, 2, \dots, Q; n = 1, 2, \dots, Q$) must be a characteristic solution of equations (83). In the latter system of equations, the matrix $R^Q(k, j)$ is symmetric in the indices k and j . It therefore follows from equations (83) that the constants $a(j, n)$ satisfy the conditions

$$\sum_{j=1}^Q a(j, m)a(j, n) = \delta_{mn}. \quad (92)$$

Without loss of generality, we have imposed here the normalization condition that, when $m = n$, the sum on the left-hand side of equation (92) is equal to unity. Equations (91) and (92) combine to establish equations (90). These are precisely of the form of the orthogonality relations that must be satisfied by the exact solutions of the characteristic value problem (see eqs.[24] in LM).

In virtue of equations (89), we can also combine equations (91) and (92) in the manner

$$\langle A\boldsymbol{\eta}(m), \boldsymbol{\eta}(n) \rangle_2 = \sum_{j=1}^Q (-1)^j \alpha(j, m)\alpha(j, n) = \delta_{mn}. \quad (93)$$

In other words, the constants $\alpha(j, n)$ are normalized in the manner

$$\sum_{j=1}^Q (-1)^j [\alpha(j, n)]^2 = 1. \quad (94)$$

5.3 Identification of modes in different approximations

We shall distinguish solutions of equations (87) obtained in different approximations with the aid of subscripts. Thus, let $\omega_a(n)$, $a_a(j, n)$, $\alpha_a(j, n)$, and $\boldsymbol{\eta}_a(n)$ denote the quantities involved in the representation of the n th characteristic solution of equations (87) in the a th approximation, where $a = k_{\max} + 1$. Accordingly, we write equation (88) for the n th characteristic solution for the Lagrangian displacement in the a th approximation as

$$\boldsymbol{\eta}_a(n) = \sum_{j=1}^{N_a} a_a(j, n)\boldsymbol{\beta}_a(j), \quad (95)$$

where $N_a = (k_{\max} + 1)(k_{\max} + 3) = a(a + 2)$. The basis vectors $\boldsymbol{\beta}_a(j)$ are labelled here with a subscript a , because the sets of basis vectors constructed in Section 4.3 are different in the different approximations. The number of characteristic solutions of equation (87) in the a th approximation is equal to $Q_a = (k_{\max} + 1)(k_{\max} + 2) = a(a + 1)$.

A useful comparison of the characteristic solutions $\boldsymbol{\eta}_a(n)$ and $\boldsymbol{\eta}_b(m)$ ($a > b$) is provided by the inner product

$$I_{ab}(n, m) = \langle A\boldsymbol{\eta}_a(n), \boldsymbol{\eta}_b(m) \rangle_2 = \sum_{j=1}^{N_a} \sum_{k=1}^{N_b} a_a(j, n)a_b(k, m) \langle A\boldsymbol{\beta}_a(j), \boldsymbol{\beta}_b(k) \rangle_2. \quad (96)$$

The values of $|I_{ab}(n, m)|$ provide criteria for identifying characteristic solutions of equations (87) that represent the same mode in different approximations. Note that we take the modulus of the inner product here, because the phases of the solutions $\boldsymbol{\eta}_a(n)$ and $\boldsymbol{\eta}_b(m)$ need not be the same. According to equation (24) in LM, the Lagrangian displacements of different modes are mutually orthogonal. If $\boldsymbol{\eta}_a(n)$ and $\boldsymbol{\eta}_b(m)$ were accurate representations of the Lagrangian displacement of the same mode, then we would have $|I_{ab}(n, m)| = 1$. Otherwise, if $\boldsymbol{\eta}_a(n)$ and $\boldsymbol{\eta}_b(m)$ were accurate representations of the Lagrangian displacements of different modes, then we would have $|I_{ab}(n, m)| = 0$. The characteristic solutions of equations (87) give only approximate representations of the Lagrangian displacements of modes, in general, so the values of $|I_{ab}(n, m)|$ should lie between 0 and 1 in practice.

Consider now the evaluation of the inner products $\langle A\boldsymbol{\beta}_a(j), \boldsymbol{\beta}_b(k) \rangle_2$ on the right-hand side of equation (96). The prescription for the construction of basis vectors described in Section 4.3 (see particularly the ordering of the vector polynomials in equations [52] and [53] there) implies that

$$\boldsymbol{\beta}_b(k) = \boldsymbol{\beta}_a(k) \quad (k = 1, 2, \dots, Q_b). \quad (97)$$

Therefore, we can reduce equation (96) to the form

$$I_{ab}(n, m) = \sum_{k=1}^{Q_b} a_a(k, n)a_b(k, m) + \sum_{j=Q_b+1}^{N_a} \sum_{k=Q_b+1}^{N_b} a_a(j, n)a_b(k, m) \langle A\boldsymbol{\beta}_a(j), \boldsymbol{\beta}_b(k) \rangle_2 \quad (98)$$

with the aid of the orthogonality relations governing the basis vectors (see eqs. [64]). Substituting from equations (57) for $\beta_b(k)$, and again making use of equations (64) and (97), we verify that

$$\langle A\beta_a(j), \beta_b(k) \rangle_2 = N_b(k) \langle A\beta_a(j), \pi_b(k) \rangle_2 \quad (Q_b < j \leq N_a; Q_b < k \leq N_b). \quad (99)$$

It also follows from equations (52) and (53) that

$$\pi_b(k) = \pi_a(l) \quad (k > Q_b; l = k + Q_a - Q_b > Q_a). \quad (100)$$

We also have

$$N_b(k) = N_a(l) = 1 + i \quad (k > Q_b; l = k + Q_a - Q_b > Q_a) \quad (101)$$

in virtue of equations (76). We can therefore reduce the right-hand side of equation (99) in the manner

$$\begin{aligned} \langle A\beta_a(j), \beta_b(k) \rangle_2 &= N_a(l) \langle A\beta_a(j), \pi_a(l) \rangle_2 \\ &= -N_a(l) \sum_{r=1}^{Q_a} c_a(l, r) \langle A\beta_a(j), \beta_a(r) \rangle_2 \quad (Q_b < j \leq N_a; Q_a < l = k + Q_a - Q_b \leq N_b + Q_a - Q_b), \end{aligned} \quad (102)$$

where we have eliminated $\pi_a(l)$ with the aid of equations (57) and we have suppressed terms that vanish identically with the aid equations (24), (64), and (75). Upon substituting from equation (102) for $\langle A\beta_a(j), \beta_b(k) \rangle_2$ on the right-hand side of equation (98) and making further use of equations (24) and (64) in the evaluation of the inner products $\langle A\beta_a(j), \beta_a(r) \rangle_2$, we finally obtain

$$I_{ab}(n, m) = \sum_{k=1}^{Q_b} a_a(k, n) a_b(k, m) - \sum_{j=Q_b+1}^{Q_a} \sum_{l=Q_a+1}^{N_b+Q_a-Q_b} a_a(j, n) a_b(l - Q_a + Q_b, m) N_a(l) c_a(l, j). \quad (103)$$

6 NUMERICAL EXAMPLES

In this section, we illustrate the matrix method with a calculation of modes of radial oscillation in a family of spherically symmetric models of galaxies. In each member of the family, the unperturbed distribution function is a function of the energy alone, and the unperturbed density is of the form

$$\rho_0(r) = \frac{3M_0 r_c^2}{4\pi(r_c^2 + r^2)^{5/2}} - \frac{3M_0 r_c^2}{4\pi(r_c^2 + R^2)^{5/2}} \quad (r \leq R), \quad (104)$$

where R is the radius of the configuration, the constant r_c is of the nature of a core radius, and the constant M_0 is a certain characteristic mass. It follows from equation (104) that the characteristic mass is determined in terms of the central density $\rho_0(0)$, the radius R , and the core radius r_c in the manner

$$M_0 = \frac{4\pi}{3} \rho_0(0) R^3 x_c^3 \left[1 - \frac{x_c^5}{(x_c^2 + 1)^{5/2}} \right]^{-1}, \quad (105)$$

where $x_c = r_c/R$. We can evaluate the mass of the system interior to a spherical surface of radius r as

$$M_0(r) = 4\pi \int_0^r \rho_0(r) r^2 dr = \frac{M_0 r^3}{(r_c^2 + r^2)^{3/2}} - \frac{M_0 r_c^2 r^3}{(r_c^2 + R^2)^{5/2}}, \quad (106)$$

and we can accordingly evaluate the gravitational acceleration in the unperturbed system with the aid of Newton's theorem. In the limit $R \rightarrow \infty$, the members of this family of models tend to Plummer's model.

We make use of R , $\rho_0(0)$, and $[\pi G \rho_0(0)]^{-1/2}$ as the units of length, density, and time, respectively. In this system of units, the models form a one-parameter sequence. The dimensionless core radius $x_c = r_c/R$ is a convenient parameter with which to distinguish the members of the sequence. Alternatively, we can make use of the central concentration, defined here as the ratio $C = \rho_0(0)/\bar{\rho}_0$ of the central density to the mean density of a configuration, as the parameter which distinguishes models along the sequence. With the aid of equations (105) and (106), it is a straight-forward calculation to verify that

$$C \equiv \frac{\rho_0(0)}{\bar{\rho}_0} = \frac{4\pi}{3} \rho_0(0) R^3 \left[4\pi \int_0^R \rho_0(r) r^2 dr \right]^{-1} = \frac{1}{x_c^3} [(x_c^2 + 1)^{5/2} - x_c^5]. \quad (107)$$

The solutions of equations (87) occur in pairs, the frequencies of each pair being equal in magnitude but opposite in algebraic signs, so it will suffice to present results of the present application of the matrix method only for modes with positive frequencies. Moreover, we consider only modes which we can identify in both the third and fourth approximations. As is explained at the end of Section 5, we make use of the values of the inner products $I_{43}(n, m)$ as criteria for the identification of such modes. For a given unperturbed system, we have 12 solutions of equations (87) in the third approximation

Table 1. Identification of modes of radial oscillation in a system in which $C = 11.335$ ($x_c = 0.55$).

Mode	n	m	l	$ I_{43}(n, m) $	$ I_{42}(n, l) $	$ I_{32}(m, l) $	$\omega_4(n)$	$\omega_3(m)$	$\omega_2(l)$
R1	19	9	5	0.9999	0.9994	0.9995	0.6143	0.6144	0.6162
R2	20	12		0.9088			0.8628	1.1280	
R3	18	11	6	0.9243	0.8448	0.7429	1.3000	1.3510	1.4180
R4	17	10		0.9938			1.6360	1.6460	
R6	15	8	4	0.8928	0.5608	0.6499	2.4700	2.6310	2.3510
R9	4	2		0.8666			4.1300	4.2430	

($k_{\max} = 2, Q = 12$), and 20 solutions in the fourth approximation ($k_{\max} = 3, Q = 20$). Thus, we can expect to identify up to six modes with positive frequencies by comparing solutions of equations (87) in the third and fourth approximations. For this purpose we have evaluated $|I_{43}(n, m)|$ for all pairs of values (n, m) ($n = 1, 2, \dots, 20; m = 1, 2, \dots, 12$), but we have arbitrarily selected for further investigation only those combinations in which $|I_{43}(n, m)| \geq 0.4$. Making similar use of the inner products $|I_{42}(n, m)|$ and $|I_{32}(n, m)|$ as criteria, we can also identify solutions of equations (87) in the second approximation which represent up to three of the modes identified in the third and fourth approximations.

Table 1 illustrates the identification of six modes in an unperturbed configuration in which $x_c \equiv r_c/R = 0.55$ and $C \equiv \rho_0(0)/\bar{\rho}_0 = 11.335$ (see eq. [107]). In the first column there, we label modes in terms of their characteristic frequencies in the fourth approximation. We arrange the positive solutions of equations (87) for the frequencies in the fourth approximation in a sequence in the order of increasing values, and we identify the n th frequency in that sequence and the mode to which it belongs with the label ‘Rn’. The integers l, m , and n in the next three columns of Table 1 identify solutions of equations (87) in the second, third, and fourth approximations, respectively, with labels which were assigned by the LANPAK subroutines used in the solution of equations (87). The inner products $I_{43}(n, m)$, $I_{42}(n, l)$, and $I_{32}(m, l)$ have been evaluated with the aid of equation (103). In the last three columns of the table, $\omega_2(l)$, $\omega_3(m)$, and $\omega_4(n)$ denote solutions for the characteristic frequencies in the second, third, and fourth approximations, respectively.

The solutions listed in Table 1 and their counterparts with negative frequencies are all of the cases in which $|I_{43}(n, m)| \geq 0.4$. The result that the listed values of $|I_{43}(n, m)|$ (values which also apply to the corresponding cases with negative frequencies) exceed this limit by such a wide margin and approach unity so closely indicates that this comparison of the solutions of equations (87) in the third and fourth approximations is a reliable basis for the identification of modes. The trends in the values of $|I_{32}(m, l)|$, $|I_{42}(n, l)|$, and $|I_{43}(n, m)|$ for the modes which are also represented in the second approximation indicate that the sequence of approximations represented in Table 1 tends to converge. The values of the characteristic frequencies listed in Table 1 in the different approximations also indicate that the identification of modes in the different approximations is reliable and that the sequence of approximations tends to converge.

For each of the modes represented in Table 1, we plot in Figures 1-6 the solutions in the second, third, and fourth approximations for the amplitudes $\alpha(k)$ and for the quantity

$$M_1(r, t) = M_1(r, 0) \exp(-i\omega t) = \int_{|\mathbf{x}| < r} \rho_1(\mathbf{x}, t) d\mathbf{x} = - \int_{|\mathbf{x}| < r} \left[\frac{\partial}{\partial \mathbf{x}} \cdot (\rho_0 \boldsymbol{\xi}) \right] d\mathbf{x} = -4\pi r^2 \rho_0(r) \frac{\mathbf{x}}{r} \cdot \boldsymbol{\xi}(\mathbf{x}, t), \quad (108)$$

which is the Eulerian perturbation of the mass interior to a spherical surface of radius r . In writing this expression for $M_1(r, t)$, we have represented the Eulerian perturbation of the density $\rho_1(\mathbf{x}, t)$ in terms of the hydrodynamical Lagrangian displacement $\boldsymbol{\xi}(\mathbf{x}, t)$ with the aid of the relations

$$\rho_1(\mathbf{x}, t) = - \frac{\partial}{\partial \mathbf{x}} \cdot (\rho_0 \boldsymbol{\xi}) \quad \text{and} \quad \rho_0(r) \boldsymbol{\xi}(\mathbf{x}, t) = m_* \int \Delta \mathbf{x}(\mathbf{x}, \mathbf{v}, t) f_0(\mathbf{x}, \mathbf{v}) d\mathbf{v} \quad (109)$$

(Vandervoort 1983). In the second of equations (109), the integration extends over the region of the velocity space accessible to stars in the neighborhood of the point \mathbf{x} . In order to evaluate $\boldsymbol{\xi}(\mathbf{x}, t)$, we express the Lagrangian displacement $\boldsymbol{\eta}$ as a superposition of the vector monomials $\boldsymbol{\mu}_\sigma^T(\mathbf{z}_{+1}, \mathbf{z}_{-1}; m, n, p)$ with the aid of equations (55), (57), and (88), we extract from that superposition an expression for $\Delta \mathbf{x}(\mathbf{x}, \mathbf{v}, t)$, and we express the result of integrating $\Delta \mathbf{x}(\mathbf{x}, \mathbf{v}, t)$ over the velocity space as a superposition of the moments $G(r, \mu)$ of the velocity distribution described in Appendix E.

It happens that we can choose the phase of the solution for each mode so that the solutions for $\alpha(k)$ are real; we adopt that choice in all of the calculations described below. A comparison of the amplitudes $\alpha(k)$ in different approximations is a direct comparison of solutions of the characteristic value problem in the reduced matrix representation described by equations (87). Implicitly, this is a comparison of Lagrangian displacements in the different approximations. With the phase of the mode chosen so that the coefficients $\alpha(k)$ are real, it turns out that $M_1(r, 0)$ is a complex quantity with equal real and imaginary parts. It will therefore suffice to plot the real part of $M_1(r, 0)$ in what follows. A comparison of the solutions for $\text{Re}[M_1(r, 0)]$ in the different approximations is a comparison of the different approximations for the structures of perturbations in the configuration space.

The mode R1 listed in Table 1 is the fundamental mode of radial oscillation. Evidently, the fundamental mode is represented very accurately in these calculations. The plots of the solutions for $\alpha(k)$ and $\text{Re}[M_1(r, 0)]$ are displayed in Figure 1.

Figure 1. Representations of the mode R1 in three approximations in a system in which $C = 11.335$ ($x_c = 0.55$). This is the fundamental mode of radial oscillation. (a) Amplitudes $\alpha(k)$ in the representation of the perturbation. (b) Dependence of the Eulerian perturbation of the mass variable $\text{Re}[M_1(r, 0)]$ on the distance r from the centre of the unperturbed system.

Figure 2. Representations of the mode R2 in two approximations in a system in which $C = 11.335$ ($x_c = 0.55$). (a) Amplitudes $\alpha(k)$ in the representation of the perturbation. (b) Dependence of the Eulerian perturbation of the mass variable $\text{Re}[M_1(r, 0)]$ on the distance r from the centre of the unperturbed system.

The mode is already well represented in the second approximation, and the additional amplitudes $\alpha(k)$ that appear in the third and fourth approximations are almost negligible. The solutions for $\text{Re}[M_1(r, 0)]$ converge rapidly.

The mode R2 is not represented very accurately in these calculations. According to Table 1, the discrepancy in the values of the frequency in the third and fourth approximations is substantial. In Figure 2a, the amplitudes $\alpha(k)$ ($12 < k \leq 20$) in the fourth approximation are not negligible, so the representations of the mode in the third and fourth approximations must be significantly different. The difference is manifest in the plots of $\text{Re}[M_1(r, 0)]$ in Figure 2b.

The convergence of the representations of mode R3 in the three approximations that is already evident in Table 1 is also evident in Figure 3. The amplitudes $\alpha(k)$ in the three approximations are in relatively good agreement, and, in the fourth approximation the amplitudes $\alpha(k)$ ($12 < k \leq 20$) are relatively small if not quite negligible. The plots of $\text{Re}[M_1(r, 0)]$ show a satisfactory convergence.

The representations of the mode R4 in Table 1 and Figure 4 appear to be quite accurate. The solutions for the amplitudes $\alpha(k)$ ($1 \leq k \leq 12$) in the third and fourth approximations agree very well, and the values of $\alpha(k)$ ($12 < k \leq 20$) in the fourth approximation are quite small. The runs of $\text{Re}[M_1(r, 0)]$ in the two approximations are remarkably similar.

The values of $|I_{32}(m, l)|$ and $|I_{42}(n, l)|$ listed in Table 1 for the mode R6 indicated that that mode is not accurately represented in the second approximation. In Figure 5(a), the amplitudes $\alpha(k)$ in the second approximation do not agree very well with their counterparts in the third and fourth approximations, and the amplitudes $\alpha(k)$ ($k > 6$) are substantial in the third and fourth approximations. Likewise, in the runs of $\text{Re}[M_1(r, 0)]$ in Figure 5(b), the second approximation does not

Figure 3. Representations of the mode R3 in three approximations in a system in which $C = 11.335$ ($x_c = 0.55$). (a) Amplitudes $\alpha(k)$ in the representation of the perturbation. (b) Dependence of the Eulerian perturbation of the mass variable $\text{Re}[M_1(r, 0)]$ on the distance r from the centre of the unperturbed system.

Figure 4. Representations of the mode R4 in two approximations in a system in which $C = 11.335$ ($x_c = 0.55$). (a) Amplitudes $\alpha(k)$ in the representation of the perturbation. (b) Dependence of the Eulerian perturbation of the mass variable $\text{Re}[M_1(r, 0)]$ on the distance r from the centre of the unperturbed system.

provide an accurate representation of the mode. Nevertheless, the agreement of the representations of the mode R6 in the third and fourth approximations is quite good.

In the plots of $\alpha(k)$ and $\text{Re}[M_1(r, 0)]$ in Figure 6, it appears that the identification of the mode R9 in Table 1 is reliable and that the representation of the mode is relatively accurate.

The preceding discussion illustrates the matrix method and identifies six of the lowest modes of radial oscillation in a system with a relatively modest central concentration ($C \equiv \rho_0(0)/\bar{\rho}_0 = 11.335$; $x_c \equiv r_c/R = 0.55$). Figure 7 shows the dependence of the characteristic periods $P = 2\pi/\omega$ on the central concentration C for four modes which can be identified in both the third and fourth approximations and traced over a range of substantially greater central concentrations. Values of the periods derived in the second, third, and fourth approximations are plotted as open triangles, circles, and squares, respectively. We have chosen to plot the periods instead of the frequencies in Figure 7 in order to facilitate comparisons of the different approximations in systems with higher central concentrations. The criterion adopted here for the identification of a mode in the third and fourth approximations is the condition $|I_{43}(n, m)| \geq 0.75$. When this criterion is satisfied, points representing the periods in the third and fourth approximations are plotted as large circles and large squares respectively. The smaller circles and squares represent extrapolations or interpolations to values of the central concentrations at which the values of $|I_{43}(n, m)|$ are less than 0.75. Two of the modes presented in Figure 7 are also represented in the second approximation, and those results are plotted as large triangles or small triangles according as $|I_{42}(n, l)| \geq 0.75$ or $|I_{42}(n, l)| < 0.75$.

In the present calculations, it has not been possible to obtain accurate solutions of equations (87) in the fourth ap-

Figure 5. Representations of the mode R6 in three approximations in a system in which $C = 11.335$ ($x_c = 0.55$). (a) Amplitudes $\alpha(k)$ in the representation of the perturbation. (b) Dependence of the Eulerian perturbation of the mass variable $\text{Re}[M_1(r, 0)]$ on the distance r from the centre of the unperturbed system.

Figure 6. Representations of the mode R9 in two approximations in a system in which $C = 11.335$ ($x_c = 0.55$). (a) Amplitudes $\alpha(k)$ in the representation of the perturbation. (b) Dependence of the Eulerian perturbation of the mass variable $\text{Re}[M_1(r, 0)]$ on the distance r from the centre of the unperturbed system.

proximation for unperturbed systems in which $C > 1025$ ($x_c < 0.1$). The failure of these calculations for the most centrally concentrated systems considered was traced to the accumulation of round-off errors in the numerical solution of equations (61)-(63) for the quantities $N(q)$ and $c(q, r)$.

The results displayed in Figure 7(a) indicate that the present representations of the fundamental mode, the mode R1, remain quite accurate in systems with central concentrations up to values at least of the order of 10,000. The behavior of the characteristic frequency in the fourth approximation is described in the most centrally concentrated systems by the formula

$$\omega \approx 3.5(\pi G \bar{\rho}_0)^{1/2} \quad (C \gg 100), \quad (110)$$

where $\bar{\rho}_0$ is the mean density of the system. For the fundamental mode in systems in which $C = 11.34$ ($x_c = 0.55$) and in which $C = 1025$ ($x_c = 0.1$) plots of $\alpha(k)$ and $\text{Re}[M_1(r, 0)]$ are shown in Figures 1 and 8, respectively. Whereas the second, third, and fourth approximations are substantially the same in the less centrally concentrated system represented in Figure 1, the three approximations are rather different in the more centrally concentrated system represented in Figure 8. Nevertheless, the sequence of approximations does appear to be converging to an accurate representation of the mode in the latter case. The node near the origin in the run of $\text{Re}[M_1(r, 0)]$ in Figure 8(b) is unexpected. That feature need not be real; it could be a result of the accumulation of numerical errors mentioned above or it could be an artifact of the present method of approximation. That this feature is absent in the second approximation, present in the third approximation, and almost absent in the fourth approximation suggest that it might be an artifact. The feature does not appear at lower central concentrations.

Figure 7. Dependence of the characteristic period $P = 2\pi/\omega$ on the central concentration C for four modes of radial oscillation. Open triangles, circles, and squares represent the second, third, and fourth approximations, respectively. The different sizes of the symbols are explained in the text. Filled circles represent the results of N-body solutions of the Lagrangian perturbation equations.

Figure 7(a) includes a comparison of the present results with values of the period of the fundamental mode calculated for the same unperturbed systems with the aid of an N-body method for the solution of the Lagrangian perturbation equations (Vandervoort 1999). The N-body results, which are represented by filled circles in Figure 7(a), were obtained before the start of the present work on the matrix method. A mode is excited in such N-body experiments by imposing suitable initial conditions on the Lagrangian displacements $(\Delta\mathbf{x}, \Delta\mathbf{v})$ of the particles. In the experiments represented in Figure 7(a), suitable initial conditions were found by a process of trial and error in which corrections were applied to virial estimates of the Lagrangian displacements (Vandervoort 1983,1999). Those efforts to excite the fundamental mode without exciting other perturbations significantly were successful only for systems in which the central concentrations are sufficiently low ($C \leq 137.8$; $x_c \geq 0.2$). In future work, it will be of interest to make use of Lagrangian displacements derived with the aid of the matrix method as initial conditions with which to excite the fundamental mode and other modes in N-body experiments on systems covering a wider range of central concentrations.

In Figure 7(b), it appears that the identification of the mode R2 is quite firm over the full range of central concentrations spanned by the plotted points. Even the smaller points plotted at $C = 17.20, 22.49$, and 31.01 represent cases in which $|I_{43}(n, l)| > 0.7$. Figure 9 shows plots of $\alpha(k)$ and $\text{Re}[M_1(r, 0)]$ for the mode R2 in the case that $C = 1025$ ($x_c = 0.1$). The plots of $\alpha(k)$ in the third and fourth approximations appear to be quite different; nevertheless the runs of $\text{Re}[M_1(r, 0)]$ in the two approximations agree rather well. The node that appears near the origin in the run of $\text{Re}[M_1(r, 0)]$ in the fourth approximation seems to be an artifact of the approximation like the similar feature mentioned above in the case of the fundamental mode. The fourth approximation to the mode R2 does not have such a feature when $C = 137.8$ ($x_c = 0.2$). A comparison of Figure 9(b) with Figure 2(b) indicates that the mode R2 has significantly different structures in systems of high and low central concentrations. The run of $\text{Re}[M_1(r, 0)]$ has two interior nodes at $C = 11.34$ ($x_c = 0.55$) but only one

Figure 8. Representations of the mode R1 in three approximations in a system in which $C = 1025$ ($x_c = 0.1$). This is the fundamental mode of radial oscillation. (a) Amplitudes $\alpha(k)$ in the representation of the perturbation. (b) Dependence of the Eulerian perturbation of the mass variable $\text{Re}[M_1(r, 0)]$ on the distance r from the centre of the unperturbed system.

Figure 9. Representations of the mode R2 in two approximations in a system in which $C = 1025$ ($x_c = 0.1$). (a) Amplitudes $\alpha(k)$ in the representation of the perturbation. (b) Dependence of the Eulerian perturbation of the mass variable $\text{Re}[M_1(r, 0)]$ on the distance r from the centre of the unperturbed system.

interior node at $C = 1025$ ($x_c = 0.1$). The transition from two nodes to one node occurs at $C \approx 13.59$ ($x_c \approx 0.5$) in the third approximation and at $C \approx 17.20$ ($x_c \approx 0.45$) in the fourth approximation.

Although it was shown above that, for the system in which $C = 11.34$ ($x_c = 0.55$), the identification of the mode R4 is reliable and the representation of the mode is accurate, the identification and representation of the mode becomes problematic at higher central concentrations. In the representation of the mode in Figure 7(c), the runs of two of the solutions for the characteristic frequency in the third approximation are shown. The application of the criterion $|I_{43}(n, m)| \geq 0.75$ implies that the lower of the two curves represents the mode R4 in the cases that $C \leq 22.49$ ($x_c \geq 0.4$) whereas the higher curve represents the mode in the cases that $31.01 \leq C \leq 137.8$ ($0.35 \geq x_c \geq 0.2$). At central concentrations exceeding 137.8, we find no solution of equation (87) in the third approximation that can be identified convincingly as a representation of the mode R4. (Of course the nomenclature adopted in order to label modes implies that the mode R4 is correctly represented in the fourth approximation by definition.) Plots of $\alpha(k)$ and $\text{Re}[M_1(r, 0)]$ are shown in Figure 10 for the mode R4 in the case that $C = 137.8$ ($x_c = 0.2$). These results in the third and fourth approximations agree well enough to support the identification of the mode in both approximations but not well enough to show that we have an accurate solution for the mode in this case. And we have no indication of the accuracy with which the mode is represented in the fourth approximation at greater central concentrations.

As is shown in Figure 7(d), the mode R8 can be identified in accordance with the criterion $|I_{43}(n, m)| \geq 0.75$ only in centrally concentrated systems. The smaller circles and squares represent points in the third and fourth approximations,

Figure 10. Representations of the mode R2 in two approximations in a system in which $C = 137.8$ ($x_c = 0.2$). (a) Amplitudes $\alpha(k)$ in the representation of the perturbation. (b) Dependence of the Eulerian perturbation of the mass variable $\text{Re}[M_1(r, 0)]$ on the distance r from the centre of the unperturbed system.

Figure 11. Representations of the mode R8 in two approximations in a system in which $C = 1025$ ($x_c = 0.1$). (a) Amplitudes $\alpha(k)$ in the representation of the perturbation. (b) Dependence of the Eulerian perturbation of the mass variable $\text{Re}[M_1(r, 0)]$ on the distance r from the centre of the unperturbed system.

respectively, at which $0.55 < |I_{43}(n, m)| < 0.75$. Likewise, the small triangles represent points on the run of the period in the second approximation at which $0.55 < |I_{42}(n, l)| < 0.75$. Plots of $\alpha(k)$ and $\text{Re}[M_1(r, 0)]$ are shown in Figure 11 for the mode R8 in the case that $C = 1025$ ($x_c = 0.1$). The solutions for the amplitudes $\alpha(k)$ in the three approximations agree quite well in Figure 11(a), but the amplitudes that are added at higher values of k in the third and fourth approximations are significant. The three approximations for the run of $\text{Re}[M_1(r, 0)]$ agree quite well in Figure 11(b). It should be noted, however, that the three approximations do not agree on the number of nodes in the outer regions of the system. The hydrodynamical Lagrangian displacement is badly determined in the outer regions of a centrally concentrated system because there is very little mass there. In particular, the (two or) three approximations often disagree on the number of interior nodes in the run of the hydrodynamical Lagrangian displacement near the outer boundary of the unperturbed system.

7 THE CONTINUOUS SPECTRUM OF NORMAL MODES

In all of the cases described in Section 6, the modes that have been identified have frequencies which lie within the continuous spectrum of the frequencies of the radial components of the stellar orbits. Specifically, the frequencies of the modes considered here lie within the interval

$$\frac{GM_0(R)}{R^3} \leq \omega^2 \leq \frac{16\pi G\rho_0}{3}. \quad (111)$$

The lower limit in this inequality is the Keplerian frequency of an orbit at the outer edge of the system, whereas the upper limit is the frequency of the radial component of a (small) simple-harmonic oscillation of a star about the centre of the system. Inequality (111) implies that the modes identified here lie within the continuous spectrum of radial modes that is to be expected in the systems considered.

In general, the normal modes of oscillation of a galaxy include a set of modes with a discrete spectrum of real and complex frequencies and a set of modes with a continuous spectrum of real frequencies. Modes of instability would belong to the discrete spectrum. The frequencies belonging to the continuous spectrum satisfy resonance conditions with the frequencies of the unperturbed stellar orbits in the system. In particular, frequencies satisfying inequality (111) would satisfy appropriate resonance conditions for a continuous spectrum of radial modes in the systems considered in the present work. Examples of systems in which the normal modes include both discrete and continuous spectra of frequencies are described by Fridman & Polyachenko (1984; see particularly Chapter II, Section 7 and Chapter III, Sections 3.4.1 & 6.1.3). Mathur (1990; see also Weinberg 1991b) has investigated conditions under which the normal modes of a system include a discrete spectrum of real frequencies which lie outside the continuous spectrum.

The question arises in the present work as to how discrete modes can appear within the continuous spectrum of modes. The probable explanation is that such discrete modes are isolated members of the continuous spectrum with frequencies which ‘accidentally’ satisfy the characteristic equation.

This interpretation is supported by an investigation of the solutions of the Lagrangian perturbation equations for the complete spectra of normal modes in stellar systems (Vandervoort 2001). That investigation is based on a matrix method for the study of perturbations which is rather different from the method described in this paper but similar to the matrix method of Kalnajs (1977). For our present purposes, the principal results are the following. For the discrete spectrum of modes in a given system, there is a characteristic equation for the determination of the frequencies. In general, however, there is no characteristic equation governing frequencies in the continuous spectrum. On the other hand, there are exceptional frequencies in the continuous spectrum which do satisfy the characteristic equation for discrete modes. Although the details of that work must be left to a later publication, the circumstances in which isolated frequencies in the continuous spectrum can satisfy the characteristic equation can be illustrated with the aid of well known results in the theory of plasma oscillations.

In the second matrix formulation of Vandervoort (2001), the modes of oscillation of a galaxy are recognized as counterparts of the van Kampen modes of oscillation in a homogeneous plasma (van Kampen 1955, 1957; Case 1959). For the sake of simplicity, we consider a model of a plasma which consists of an electron gas of uniform density in the presence of a fixed, uniform background of positive charge. A van Kampen mode is an electrostatic plane wave of wave number k and frequency ω , say. The simplest example is obtained for a one-dimensional representation in which the electrostatic field is parallel to the direction of the wave, and functions of the velocity of an electron are integrated over the components of the velocity perpendicular to the direction of the wave. The Eulerian perturbation of the distribution function of the electrons and the electrostatic field are determined by solving Vlasov’s equation (i.e., the linearized, collisionless Boltzmann equation) and Poisson’s equation simultaneously. The Eulerian perturbation of the distribution function is of the form $f_1(x, v, t) = g(v)\exp[i(kx - \omega t)]$ where the x -direction is the direction of the wave and the amplitude $g(v)$ is a function of the x -component v of the velocity. The solution for $g(v)$ is of the form

$$g(v) = u_P^2 \text{P} \frac{df_0(v)}{dv} \frac{1}{u - v} + \lambda \delta(u - v), \quad (112)$$

where $f_0(v)$ is the distribution function of the unperturbed electron gas, $u = \omega/k$ is the phase velocity of the wave, $u_P = \omega_P/k$ is a characteristic velocity derived from the plasma frequency ω_P , the symbol P denotes the Cauchy principal value, $\delta(u - v)$ is Dirac’s delta function, and λ is a constant to be determined. Applying the normalization condition $\int g(v)dv = 1$ to equation (112), we obtain

$$\lambda = 1 - u_P^2 \text{P} \int_{-\infty}^{\infty} \frac{df_0(v)}{dv} \frac{1}{u - v} dv. \quad (113)$$

For an arbitrary choice of the value of $u = \omega/k$, in general, equation (113) determines the value of λ . Thus, in general, there is no dispersion relation (characteristic equation) for the determination of the frequencies ω of the van Kampen modes in the continuous spectrum. For isolated frequencies, however, it can happen that $\lambda = 0$. Such frequencies can be determined by setting the right-hand side of equation (113) equal to zero. The condition that the right-hand side of equation (113) vanish is the dispersion relation originally derived by Vlasov (1945) for plasma oscillations. It has been remarked (Bohm & Gross 1949, Bernstein, Greene & Kruskal 1957, Jackson 1960) that Vlasov’s dispersion relation picks out the frequencies of waves in which there are no electrons trapped in the electrostatic field.

In precisely the same way, isolated members of the continuous spectrum of modes in a galaxy are found to satisfy the characteristic equation for discrete modes. It is commonly expected that such perturbations would suffer Landau damping. However, Landau damping should not occur in individual modes in a spherical system in which the unperturbed distribution function depends only on the energy of a star. According to a theorem of Antonov (1961; Fridman & Polyachenko 1984;

Binney & Tremaine 1987; Palmer 1994), such a system will be stable if the distribution function is a monotonically decreasing function of the energy, as it would be in the systems considered in Section 6. Moreover, a lemma proved in LM shows that the frequencies of normal modes in a stellar system occur, in general, in complex-conjugate pairs. Thus, the frequencies of the modes considered in this paper must be real. In general, Landau damping does not occur in modes of the Van Kampen type. On the other hand, perturbations which are superpositions of modes belonging to the continuous spectrum would be expected to suffer Landau damping (Van Kampen 1955, 1957; Case 1959; see also Appendix 5.A. in Binney & Tremaine 1987). Our concentration on a normal mode analysis puts the study of Landau damping beyond the scope of the present investigation. The present calculations give no evidence for the existence in the systems studied of a discrete spectrum of radial modes with frequencies lying outside the continuous spectrum.

In the N-body experiments on the fundamental mode of radial oscillation that are represented in Figure 7(a), the orbits that are populated with stars have frequencies of the radial motions which do not span the entire range described by inequality (111). These experiments were performed on systems of 1000 bodies. The density of stars is so low in the regions of the phase space accessible to orbits with the lowest frequencies that these N-body realizations of the systems studied do not include such stars. In other words, the N-body realizations of these systems do not include stars that would resonate with the fundamental mode. Thus the fundamental mode appears to be a discrete mode in the N-body experiments, and that perturbation should not suffer Landau damping. For substantially larger (and more realistic) numbers of bodies, N-body realizations would include stars that resonate with the frequency of the fundamental mode, and perturbations that include the fundamental mode could suffer Landau damping. The damping would be expected to be slow, because the density of resonant stars in the phase space would be very low.

8 CONCLUDING REMARKS

For the study of small perturbations in stellar systems, the matrix method described in this paper has two particularly attractive features. The first is that the characteristic value problem for modes in the matrix representation is linear in the frequencies of the modes. Consequently, the solution of the characteristic equation for the frequencies is readily obtained with the aid of standard matrix methods. In contrast, the matrix method of Kalnajs involves a characteristic equation which is nonlinear in the frequencies, and it is a more challenging task to obtain the solutions for the frequencies. The second feature that makes the present matrix method particularly attractive is that the underlying Lagrangian representation deals directly with the perturbations of the trajectories of the particles. The comparison of analytic investigations of small perturbations with N-body experiments is straight forward in this case. Analytic solutions for the Lagrangian displacements of the perturbations can be used as initial conditions for N-body experiments which provide direct tests of analytic results. The experimental results included in Figure 7(a) represent such a test.

The results obtained in Section 6 consist of relatively accurate representations of six modes at central concentrations of the order of 10, quite an accurate representation of the fundamental mode of radial oscillation at central concentrations ranging up to the order of 10,000, and an approximate delineation of three other modes at central concentrations ranging up to the order of 100-1,000. These results were obtained with the aid of a very modest computational effort. Therefore, considerable scope remains for more diverse investigations of modes of oscillation and instability in stellar systems with the aid of more extensive numerical calculations.

For these reasons, the planned extension of the present investigation to nonradial modes in spheres and to modes in axisymmetric systems, as described at the end of Section 1, would appear to hold considerable promise.

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APPENDIX A: LIST OF VECTOR POLYNOMIALS

In this appendix, we list additional groups of vector polynomials identified with the aid of the procedure described at the beginning of Section 4.2.

The vector polynomials in the case that $k = 1$ are

$$\begin{aligned}
 \pi(1, 1) &= \mu_{+1}^{+1}(1, 0, 0), & \pi(1, 2) &= \mu_{-1}^{+1}(0, 0, 1), \\
 \pi(1, 3) &= \mu_{+1}^{+1}(0, 1, 0) + \frac{1}{2}\mu_{-1}^{-1}(1, 0, 0) - \frac{1}{2}\mu_{+1}^{-1}(1, 0, 0), & \pi(1, 4) &= \mu_{-1}^{+1}(0, 1, 0) + \frac{1}{2}\mu_{+1}^{-1}(0, 0, 1) - \frac{1}{2}\mu_{-1}^{-1}(0, 0, 1), \quad \text{and} \quad (A1) \\
 \pi(1, 5) &= \mu_{+1}^{+1}(0, 0, 1) - \mu_{-1}^{+1}(1, 0, 0) + 2\mu_{-1}^{-1}(0, 1, 0) - 2\mu_{+1}^{-1}(0, 1, 0),
 \end{aligned}$$

where we suppress the arguments \mathbf{z}_{+1} and \mathbf{z}_{-1} here and in what follows.

Likewise, the vector polynomials in the case that $k = 2$ are

$$\begin{aligned}
 \pi(2, 1) &= \mu_{+1}^{+1}(2, 0, 0), & \pi(2, 2) &= \mu_{-1}^{+1}(0, 0, 2), \\
 \pi(2, 3) &= \mu_{+1}^{+1}(1, 1, 0) + \frac{1}{4}\mu_{-1}^{-1}(2, 0, 0) - \frac{1}{4}\mu_{+1}^{-1}(2, 0, 0), & \pi(2, 4) &= \mu_{-1}^{+1}(0, 1, 1) + \frac{1}{4}\mu_{+1}^{-1}(0, 0, 2) - \frac{1}{4}\mu_{-1}^{-1}(0, 0, 2), \\
 \pi(2, 5) &= \mu_{+1}^{+1}(1, 0, 1) + 2\mu_{+1}^{+1}(0, 2, 0) + 2\mu_{-1}^{-1}(1, 1, 0) - \frac{1}{2}\mu_{-1}^{+1}(2, 0, 0) - 2\mu_{+1}^{-1}(1, 1, 0), & & (A2) \\
 \pi(2, 6) &= \mu_{-1}^{+1}(1, 0, 1) + 2\mu_{+1}^{+1}(0, 2, 0) + 2\mu_{+1}^{-1}(0, 1, 1) - \frac{1}{2}\mu_{+1}^{+1}(0, 0, 2) - 2\mu_{-1}^{-1}(0, 1, 1), \quad \text{and} \\
 \pi(2, 7) &= \mu_{+1}^{+1}(0, 1, 1) + \frac{1}{2}\mu_{-1}^{-1}(1, 0, 1) + \mu_{-1}^{-1}(0, 2, 0) - \mu_{-1}^{+1}(1, 1, 0) - \frac{1}{2}\mu_{+1}^{-1}(1, 0, 1) - \mu_{+1}^{-1}(0, 2, 0).
 \end{aligned}$$

Finally, the vector polynomials in the case that $k = 3$ are

$$\begin{aligned}
\pi(3, 1) &= \mu_{+1}^{+1}(3, 0, 0), & \pi(3, 2) &= \mu_{-1}^{+1}(0, 0, 3), \\
\pi(3, 3) &= \mu_{+1}^{+1}(2, 1, 0) + \frac{1}{6}\mu_{-1}^{-1}(3, 0, 0) - \frac{1}{6}\mu_{+1}^{-1}(3, 0, 0), & \pi(3, 4) &= \mu_{-1}^{+1}(0, 1, 2) + \frac{1}{6}\mu_{+1}^{-1}(0, 0, 3) - \frac{1}{6}\mu_{-1}^{-1}(0, 0, 3), \\
\pi(3, 5) &= \mu_{+1}^{+1}(2, 0, 1) + 4\mu_{+1}^{+1}(1, 2, 0) + 2\mu_{-1}^{-1}(2, 1, 0) - \frac{1}{3}\mu_{-1}^{+1}(3, 0, 0) - 2\mu_{+1}^{-1}(2, 1, 0), \\
\pi(3, 6) &= \mu_{-1}^{+1}(1, 0, 2) + 4\mu_{+1}^{+1}(0, 2, 1) + 2\mu_{+1}^{-1}(0, 1, 2) - \frac{1}{3}\mu_{+1}^{+1}(0, 0, 3) - 2\mu_{-1}^{-1}(0, 1, 2), \\
\pi(3, 7) &= \mu_{+1}^{+1}(1, 1, 1) + \frac{2}{3}\mu_{+1}^{+1}(0, 3, 0) + \frac{1}{4}\mu_{-1}^{-1}(2, 0, 1) + \mu_{-1}^{-1}(1, 2, 0) - \frac{1}{2}\mu_{-1}^{+1}(2, 1, 0) - \frac{1}{4}\mu_{+1}^{-1}(2, 0, 1) - \mu_{+1}^{-1}(1, 2, 0), \\
\pi(3, 8) &= \mu_{-1}^{+1}(1, 1, 1) + \frac{2}{3}\mu_{-1}^{+1}(0, 3, 0) + \frac{1}{4}\mu_{+1}^{-1}(1, 0, 2) + \mu_{+1}^{-1}(0, 2, 1) - \frac{1}{2}\mu_{+1}^{+1}(0, 1, 2) - \frac{1}{4}\mu_{-1}^{-1}(1, 0, 2) - \mu_{-1}^{-1}(0, 2, 1), \\
&\text{and} \\
\pi(3, 9) &= \mu_{+1}^{+1}(1, 0, 2) + 4\mu_{+1}^{+1}(0, 2, 1) + 4\mu_{-1}^{-1}(1, 1, 1) + \frac{8}{3}\mu_{-1}^{-1}(0, 3, 0) \\
&\quad - \mu_{-1}^{+1}(2, 0, 1) - 4\mu_{-1}^{+1}(1, 2, 0) - 4\mu_{+1}^{-1}(1, 1, 1) - \frac{8}{3}\mu_{+1}^{-1}(0, 3, 0).
\end{aligned} \tag{A3}$$

APPENDIX B: INNER PRODUCTS OF VECTOR POLYNOMIALS AND THEIR ADJOINTS

The construction of the basis vectors described in Section 4.3 requires the evaluation of the inner products $\langle A\pi(q'), \pi(q) \rangle_2$. These can be expressed in terms of the inner products $\langle A\mu_{\sigma'}^{\tau'}(m', n', p'), \mu_{\sigma}^{\tau}(m, n, p) \rangle_2$.

Making use of equations (42) and (50) for the vector monomials and their adjoint vectors, respectively, constructing the inner products of those vectors in accordance with equation (9), and taking the complex conjugates of quantities in accordance with equations (49), we obtain

$$\langle A\mu_{\sigma'}^{\tau'}(m', n', p'), \mu_{\sigma}^{\tau}(m, n, p) \rangle_2 = \text{in}_0(\tau'\sigma' + \tau\sigma)m_* \int_{\Omega} \mathbf{z}_{-\sigma'} \cdot \mathbf{z}_{\sigma} M(m + p', n + n', p + m') f_0 d\mathbf{x} d\mathbf{v}. \tag{B1}$$

By inspection of equation (B1), we see that

$$\langle A\mu_{\sigma'}^{\tau'}(m', n', p'), \mu_{\sigma}^{\tau}(m, n, p) \rangle_2 = 0 \quad \text{unless } \tau'\sigma' = \tau\sigma. \tag{B2}$$

We also note that the quantity $\mathbf{z}_{-\sigma'} \cdot \mathbf{z}_{\sigma} M(m + p', n + n', p + m')$ can be expressed as a polynomial in the components of \mathbf{x} and $i\mathbf{v}$ in virtue of equations (33) and (35). For systems of the kind considered in Section 4, in which $f_0(\mathbf{x}, \mathbf{v})$ is an even function of \mathbf{v} , it follows that the imaginary part of the integrand on the right-hand side of equation (B1) is an odd function of the components of \mathbf{v} . Thus the integral on the right-hand side of equation (B1) is a real quantity, and we have the result that $\langle A\mu_{\sigma'}^{\tau'}(m', n', p'), \mu_{\sigma}^{\tau}(m, n, p) \rangle_2$ is an imaginary quantity for systems in which the distribution function f_0 is an even function of the components of the velocity. Finally, we verify that

$$\langle A\mu_{\sigma}^{\tau}(m, n, p), \mu_{\sigma'}^{\tau'}(m', n', p') \rangle_2 = \langle A\mu_{\sigma'}^{\tau'}(m', n', p'), \mu_{\sigma}^{\tau}(m, n, p) \rangle_2. \tag{B3}$$

For, as we have just explained, the integral on the right-hand side of equation (B1) is real. Therefore, we may replace the integrand there with its complex conjugate, rewrite the result with the aid of the first two of equations (49) and, in virtue of equation (B1), transform an expression for the right-hand side of equation (B3) into an expression for the left-hand side of equation (B3).

It follows from equations (55) and (56) that

$$\langle A\pi(q'), \pi(q) \rangle_2 = \sum_{j'=1}^{J'} \sum_{j=1}^J \langle q'|j' \rangle \langle q|j \rangle \langle A\mu_{\sigma'}^{\tau'}(m', n', p'), \mu_{\sigma}^{\tau}(m, n, p) \rangle_2, \tag{B4}$$

where the values $J = J(q)$, $\sigma = \sigma(q; j)$, $\tau = \tau(q; j)$, $m = m(q; j)$, $n = n(q; j)$, $p = p(q; j)$, $J' = J(q')$, $\sigma' = \sigma(q'; j')$, $\tau' = \tau(q'; j')$, $m' = m(q'; j')$, $n' = n(q'; j')$, and $p' = p(q'; j')$ are determined by inspection of equations (45) and (A1)-(A3), as we have explained following equation (55).

The inner products $\langle A\pi(q'), \pi(q) \rangle_2$ have the following properties.

(i) Inasmuch as the coefficients $\langle q'|j' \rangle$ and $\langle q|j \rangle$ are real constants in equation (B4) and the inner products $\langle A\mu_{\sigma'}^{\tau'}(m', n', p'), \mu_{\sigma}^{\tau}(m, n, p) \rangle_2$ are imaginary quantities, the inner products $\langle A\pi(q'), \pi(q) \rangle_2$ are imaginary quantities.

(ii) Rewriting equation (B4) with the aid of equation (B3), we readily verify that $\langle A\pi(q'), \pi(q) \rangle_2$ has the symmetry

$$\langle A\pi(q'), \pi(q) \rangle_2 = \langle A\pi(q), \pi(q') \rangle_2 \quad (q' = 1, 2, \dots, N; q = 1, 2, \dots, N). \tag{B5}$$

(iii) A consequence of the first of equations (54) is that

$$\langle A\pi(q'), \pi(q) \rangle_2 = \langle A\pi(q' - 1), \pi(q - 1) \rangle_2^* \quad \text{and} \quad \langle A\pi(q'), \pi(q - 1) \rangle_2 = \langle A\pi(q' - 1), \pi(q) \rangle_2^* \quad (B6)$$

$$(q' = 2, 4, \dots, Q; q = 2, 4, \dots, Q),$$

inasmuch as complex conjugation commutes with the operator A .

(iv) Likewise, it follows from the second of equations (54) that

$$\langle A\pi(q'), \pi(q) \rangle_2 = 0 \quad (q' = Q + 1, Q + 2, \dots, N; q = Q + 1, Q + 2, \dots, N). \quad (B7)$$

More specifically, the second of equations (54) implies that the inner product on the left-hand side of equation (B7) is equal to its complex conjugate and is therefore real. However, the general result described in item (i) above implies that this inner product is imaginary. Therefore, the inner product described in equation (B7) must vanish. Equation (B7) is essentially a set of orthogonality relations for the purely imaginary vector polynomials and their adjoint vectors. In the case $q' = q$, we see that the inner product of a purely imaginary vector polynomial and its own adjoint vector vanishes. In other words, the purely imaginary vector polynomials are null vectors. It is for this reason that some of the basis vectors constructed in Section 4.3 turn out to be null vectors as well.

(v) A further consequence of equations (54) is that

$$\langle A\pi(q'), \pi(q) \rangle_2 = \langle A\pi(q'), \pi(q - 1) \rangle_2 \quad (q' = Q + 1, Q + 2, \dots, N; q = 2, 4, \dots, Q). \quad (B8)$$

For, when we substitute from equations (54) for the vector polynomials on the left-hand side of equation (B8) and recall that complex conjugation and A are commuting operators and that these inner products are imaginary quantities, we obtain the inner product on the right-hand side of equation (B8).

(vi) Finally, we observe that

$$\langle A\pi(q), \pi(q - 1) \rangle_2 = \langle A\pi(q - 1), \pi(q) \rangle_2 = 0 \quad (q = 2, 4, \dots, Q). \quad (B9)$$

The first of the equalities in equation (B9) is a special case of equation (B5). When we combine that symmetry relation with the result of putting $q' = q$ in the second of equations (B6), we find that the inner products in equation (B9) must be real. On the other hand, such inner products have been shown to be imaginary, in general, so they must vanish as claimed in equation (B9). Equation (B9) is essentially a set of orthogonality relations for the complex-conjugate pairs of vector polynomials and their adjoint vectors.

Our procedure for the evaluation of the integral over the phase space on the right-hand side of equation (B1) is described in Appendix E below. The numerical values of the inner products $\langle A\pi(q'), \pi(q) \rangle_2$ derived with the aid of equation (B4) have been checked with the aid of equations (B5)-(B9).

APPENDIX C: CERTAIN IDENTITIES

The arrangement of the basis vectors $\beta(q)$ ($0 < q \leq Q$) in complex-conjugate pairs as described in Section 4.3.1 (see eq. [68]) is a consequence of the identities presented in equation (69). In what follows, we shall prove equations (69) by induction.

We begin the proof by reducing the second of equations (65) and equation (67) in the case that $q = 2$ to

$$c(2, 1) = -N(1)\langle A\pi(1), \pi(2) \rangle_2 = 0 \quad \text{and} \quad N^2(2) = [\langle A\pi(2), \pi(2) \rangle_2 - c^2(2, 1)]^{-1} = [\langle A\pi(1), \pi(1) \rangle_2^*]^{-1} = [N^2(1)]^*, \quad (C1)$$

respectively, with the aid of equations (54) and (B9) and the first of equations (65). It follows from the first of equations (C1) that we can reduce equation (66) in the case that $s = 2$ to

$$c(q, 2) = N(2) [c(2, 1)c(q, 1) - \langle A\pi(2), \pi(q) \rangle_2] = -N(2)\langle A\pi(2), \pi(q) \rangle_2 \quad (q = 3, 4, \dots, Q). \quad (C2)$$

Equations (B6) and the second of equations (C1) (see also eqs. [65]) enable us to derive the relations

$$c(q, 2) = -N^*(1)\langle A\pi(1), \pi(q - 1) \rangle_2^* = c^*(q - 1, 1) \quad \text{and} \quad c(q, 1) = -N^*(2)\langle A\pi(2), \pi(q - 1) \rangle_2^* = c^*(q - 1, 2) \quad (C3)$$

$$(q = 4, 6, \dots, Q)$$

from equation (C2). Equations (C1) establish the first two of equations (69) in the case that $q = 2$, and equations (C3) establish the third and fourth of equations (69) in the case that $q = 4, 6, \dots, Q$ and $s = 2$.

Continuing the proof of equations (69), we now consider equation (66) for given, even values of q and s , we recall that the vector polynomials satisfy equations (B6), and we assume, in the sense of a proof by induction, that $N(s)$, $c(s, r)$, and $c(q, r)$ ($r < s < q$) satisfy equations (69). Under such conditions, we can rewrite equation (66) in the manner

$$c(q, s) = N^*(s - 1) \left[\sum_{r=1}^{s-2} c^*(s - 1, r)c^*(q - 1, r) - \langle A\pi(s - 1), \pi(q - 1) \rangle_2^* \right] \quad (C4)$$

$$= c^*(q - 1, s - 1) \quad (q = 6, 8, \dots, Q; s = 4, 6, \dots, q - 2).$$

In this reduction, we break the sum on the right-hand side of equation (66) into sums over even and odd values of r , rewrite the separate sums with the aid of the third and fourth of equations (69), suppress $c(q, q-1)$ in accordance with the second of equations (69), and recombine the results in the single sum shown in equation (C4). In the verification of this derivation, it is helpful to note that the application of the third and fourth of equations (69) turns sums over even and odd values of r into sums over odd and even values of $r-1$, respectively. Under the same conditions and with similar reductions, we can also rewrite equation (66) in the manner

$$c(q, s-1) = N^*(s) \left[\sum_{r=1}^{s-1} c^*(s, r) c^*(q-1, r) - \langle A\pi(s), \pi(q-1) \rangle_2^* \right] \quad (C5)$$

$$= c^*(q-1, s) \quad (q = 6, 8, \dots, Q; s = 4, 6, \dots, q-2)$$

and, in the case that $s = q-1$, in the manner

$$c(q, q-1) = N(q-1) \sum_{r=1}^{q-2} c(q-1, r) c(q, r) \quad (C6)$$

$$= N(q-1) \sum_{r=1}^{q-2} c^*(q, r) c^*(q-1, r) = 0 \quad (q = 4, 6, \dots, Q).$$

The quantity $c(q, q-1)$ must vanish here, because, the equality of the two sums in equation (C6) implies, on the one hand that each sum is a real quantity, whereas our discussion just prior to Section 4.3.1 shows, on the other hand, that the terms $c(q-1, r) c(q, r)$ and $c^*(q, r) c^*(q-1, r)$ in those sums must be imaginary quantities. Finally, we can transform the right-hand side of equation (67) with the aid of equations (B6) and (69) and derive

$$N^2(q) = \left(\langle A\pi(q-1), \pi(q-1) \rangle_2^* - \sum_{r=1}^{q-2} [c^2(q-1, r)]^* \right)^{-1} = [N^2(q-1)]^* \quad (q = 4, 6, \dots, Q). \quad (C7)$$

The completion of the proof of equations (69) now consists of starting with the results in equations (C1) and (C3), considering equations (C4)-(C7) in the sequence $q = 3, 4, \dots, Q$, and, for each value of q , verifying equations (C4) and (C5) in the sequence $s = 2, 3, \dots, q-2$ and then verifying equations (C6) and (C7). This confirms the arrangement of the basis vectors $\beta(q)$ ($0 < q \leq Q$) in complex-conjugate pairs as described in equation (68).

The construction of the null basis vectors in Section 4.3.2 requires the reduction of a subset of equations (61) and (62) to equations (72)-(74). We conclude this appendix with the derivation of identities that are required for that reduction. We must first prove that the quantities $c(q, s)$ which satisfy equations (70) and (71) also satisfy the identities

$$c(q, s) = -c^*(q, s-1) \quad (q = Q+1, Q+2, \dots, N; s = 2, 4, \dots, Q). \quad (C8)$$

For a proof of equation (C8) by induction, we first consider equation (71) in the case that $s = 2$, and we make use of equations (54), (C1), and (70) in order to write

$$c(q, 2) = -N(2) \langle A\pi(2), \pi(q) \rangle_2 = N^*(1) \langle A\pi(1), \pi(q) \rangle_2^* = -c^*(q, 1) \quad (q = Q+1, Q+2, \dots, N), \quad (C9)$$

which proves the identity in the case that $s = 2$. In order to complete the proof, we must show that the condition $c(q, r) = -c^*(q, r-1)$ holds for $r = s$, where $s(> 2)$ is an even integer, if that condition holds for even values of r which are less than s . For this purpose, we make use of equations (54) and (69) in order to reduce equation (71) to

$$c(q, s) = -N^*(s-1) \left[\sum_{r=1}^{s-2} c^*(s-1, r) c^*(q, r) - \langle A\pi(s-1), \pi(q) \rangle_2^* \right] \quad (C10)$$

$$= -c^*(q, s-1) \quad (q = Q+1, Q+2, \dots, N; s = 4, 6, \dots, Q).$$

In this reduction, we break the sum on the right-hand side of equation (71) into sums over even and odd values of r , rewrite the separate sums with the aid of the third and fourth of equations (69) and the condition $c(q, r) = -c^*(q, r-1)$ ($r = 2, 4, \dots, s-2$), make use of the second of equations (69) in order to suppress $c(s, s-1)$, and recombine the results in the single sum shown in equation (C10). That the transformed expression for $c(q, s)$ in equation (C10) is equal to $-c^*(q, s-1)$ is also a consequence of equation (71).

The identity that is required for the derivation of equations (72)-(74) is

$$\sum_{r=1}^Q c(s, r) c(q, r) = \sum_{r=1}^Q c^*(s, r) c^*(q, r) = 0 \quad (q = Q+2, Q+3, \dots, N; s = Q+1, Q+2, \dots, q). \quad (C11)$$

In this reduction, we break the first sum in equation (C11) into sums over even and odd values of r , rewrite the separate sums with the aid of equations (C8), and recombine the results in the second sum shown in equation (C11). The two sums

in equation (C11) must vanish, because, their equality implies, on the one hand, that each sum is a real quantity, whereas our earlier discussion of equations (61)–(63) shows, on the other hand, that the terms $c(s, r)c(q, r)$ and $c^*(s, r)c^*(q, r)$ in those sums must be imaginary quantities.

APPENDIX D: THE MATRIX OF THE OPERATOR P

In order to evaluate the matrix $\langle A\beta(q'), P\beta(q) \rangle_2$, we apply the operator P to the right-hand sides of equations (57), and we form the inner product of $A\beta(q')$ with the resulting expressions. We obtain

$$\langle A\beta(q'), P\beta(1) \rangle_2 = N(1) \langle A\beta(q'), P\pi(1) \rangle_2 \quad (D1)$$

and

$$\langle A\beta(q'), P\beta(q) \rangle_2 = N(q) \left[\sum_{r=1}^{q-1} c(q, r) \langle A\beta(q'), P\beta(r) \rangle_2 + \langle A\beta(q'), P\pi(q) \rangle_2 \right] \quad (q > 1). \quad (D2)$$

In order to evaluate the right-hand sides of equations (D1) and (D2), we form the inner products of the adjoint vectors $A\beta(q')$, expressed as in equations (59), with the quantities $P\pi(q)$. We obtain

$$\langle A\beta(1), P\pi(q) \rangle_2 = N(1) \langle A\pi(1), P\pi(q) \rangle_2 \quad (D3)$$

and

$$\langle A\beta(q'), P\pi(q) \rangle_2 = N(q') \left[\sum_{r'=1}^{q'-1} c(q', r') \langle A\beta(r'), P\pi(q) \rangle_2 + \langle A\pi(q'), P\pi(q) \rangle_2 \right] \quad (q' > 1). \quad (D4)$$

Finally, we construct the matrix

$$\langle A\pi(q'), P\pi(q) \rangle_2 = \sum_{j'=1}^{J'} \sum_{j=1}^J \langle q' | j' \rangle \langle q | j \rangle \langle A\mu_{\sigma'}^{\tau'}(m', n', p'), P\mu_{\sigma}^{\tau}(m, n, p) \rangle_2 \quad (D5)$$

(see eqs. [55] and [B4]) in order to evaluate the right-hand sides of equations (D3) and (D4).

Evidently, the calculations involved in the evaluation of $\langle A\beta(q'), P\beta(1) \rangle_2$ begin with the evaluation of the matrix $\langle A\mu_{\sigma'}^{\tau'}(m', n', p'), P\mu_{\sigma}^{\tau}(m, n, p) \rangle_2$ and continue with the evaluation of $\langle A\pi(q'), P\pi(q) \rangle_2$. We next consider equations (D3) and (D4) in the sequence $q = 1, 2, \dots$, and, for each value of q in turn, evaluate the quantities $\langle A\beta(q'), P\pi(q) \rangle_2$ in the sequence $q' = 1, 2, \dots$. At each step, the right-hand side of equation (D3) or (D4) can be evaluated in terms of quantities whose values are known results of preceding steps in the procedure. Likewise, we finally consider equations (D1) and (D2) in the sequence $q = 1, 2, \dots$, and, for each value of q in turn, evaluate the quantities $\langle A\beta(q'), P\beta(q) \rangle_2$ in the sequence $q' = 1, 2, \dots$. Similarly, the right-hand side of equation (D1) or (D2) can be evaluated at each step in terms of quantities whose values are known results of preceding steps.

We have now reduced numerical work required for the construction of the matrix $\langle A\beta(q'), P\beta(q) \rangle_2$ to the evaluation of the matrix

$$\begin{aligned} \langle A\mu_{\sigma'}^{\tau'}(m', n', p'), P\mu_{\sigma}^{\tau}(m, n, p) \rangle_2 = & -i\tau'\sigma'\tau\sigma n_0^2 m_* \int_{\Omega} \mathbf{z}_{-\sigma'} \cdot \mathbf{z}_{\sigma} M(m+p', n+n', p+m') f_0 d\mathbf{x} d\mathbf{v} \\ & -i(\tau'\sigma' + \tau\sigma) n_0 m_* \int_{\Omega} M(p', n', m') \mathbf{z}_{-\sigma'} \cdot D[\mathbf{z}_{\sigma} M(m, n, p)] f_0 d\mathbf{x} d\mathbf{v} \\ & + im_* \int_{\Omega} M(p', n', m') \mathbf{z}_{-\sigma'} \cdot \Delta \mathbf{a} \{ \mathbf{z}_{\sigma} M(m, n, p) \} f_0 d\mathbf{x} d\mathbf{v}. \end{aligned} \quad (D6)$$

In the derivation of equation (D6), we have constructed $\langle A\mu_{\sigma'}^{\tau'}(m', n', p'), P\mu_{\sigma}^{\tau}(m, n, p) \rangle_2$ in accordance with the definition of an inner product given in equation (9), we have substituted from equations (50), (6), and (42) for $A\mu_{\sigma'}^{\tau'}(m', n', p')$, P , and $\mu_{\sigma}^{\tau}(m, n, p)$, respectively, in the integral to be evaluated, and we have reduced equation (D6) to the form given here with the aid of equations (35) and (49).

The evaluation of the third integral on the right-hand side of equation (D6) requires the construction of the quantity $\Delta \mathbf{a} \{ \mathbf{z}_{\sigma} M(m, n, p) \}$. Let $\rho_0(r)$, $M_0(r)$, and $V_0(r)$ denote the density, the mass interior to a spherical surface of radius r , and the gravitational potential, respectively in the unperturbed system. By definition and by Newton's theorem, we have

$$M_0(r) = 4\pi \int_0^r \rho_0(r') r'^2 dr' \quad \text{and} \quad -\frac{\partial V_0}{\partial \mathbf{x}} = -\frac{GM_0(r)\mathbf{x}}{r^3}, \quad (D7)$$

respectively.

For the radial oscillations of a sphere, it is convenient to construct the Eulerian perturbation of the gravitational acceleration in terms of the Eulerian perturbation of the density $\rho_1(\mathbf{x}, t)$ and the hydrodynamical Lagrangian displacement $\boldsymbol{\xi}(\mathbf{x}, t)$ (see equations [109]). The perturbation ρ_1 is spherically symmetric, so, by Newton's theorem and then Gauss' theorem, the Eulerian perturbation of the gravitational acceleration reduces to

$$-\frac{\partial V_1}{\partial \mathbf{x}} = \frac{\mathbf{x}}{r^3} G \int_{|\mathbf{x}| < r} \left[\frac{\partial}{\partial \mathbf{x}} \cdot (\rho_0 \boldsymbol{\xi}) \right] d\mathbf{x} = \frac{\mathbf{x}}{r^3} 4\pi r^2 G \rho_0(r) \frac{\mathbf{x}}{r} \cdot \boldsymbol{\xi}(\mathbf{x}, t) = 4\pi G m_* \int \Delta \mathbf{x}(\mathbf{x}, \mathbf{v}, t) f_0(\mathbf{x}, \mathbf{v}) d\mathbf{v}. \quad (\text{D8})$$

In the reduction of equation (D8), we have noted that $\boldsymbol{\xi}$ lies in the radial direction under conditions of spherical symmetry, we have identified \mathbf{x}/r as a unit vector in the radial direction, and we have substituted from the second of equations (109) for $\rho_0 \boldsymbol{\xi}$.

Starting with the definition of the operator $\Delta \mathbf{a}\{\}$ in equation (3), we insert the Eulerian perturbation of the gravitational potential in accordance with equation (29), we make use of equations (D7) and (D8) for $-\partial V_0/\partial \mathbf{x}$ and $-\partial V_1/\partial \mathbf{x}$, respectively, and we find, after additional reductions which are straight-forward, that

$$\Delta \mathbf{a}\{z_\sigma M(m, n, p)\} = -\frac{GM_0(r)}{r^3} z_\sigma M(m, n, p) - \frac{\mathbf{x}}{r} \left[\frac{\partial}{\partial r} \left(\frac{GM_0(r)}{r^3} \right) \right] \mathbf{x} \cdot z_\sigma M(m, n, p) + 4\pi G m_* \int z_\sigma M(m, n, p) f_0 d\mathbf{v}. \quad (\text{D9})$$

APPENDIX E: INTEGRALS OVER THE PHASE SPACE

The applications of the matrix method described in Sections 5 and 6 of this paper are limited to systems in which the unperturbed distribution function is a function only of the energy of a star and the velocity distribution is accordingly isotropic. Consider in that case the integrals over the velocity space on the right-hand side of equation (D9) and within the integrals over the phase space on the right-hand sides of equations (B1) and (D6). The integrands can be expressed as polynomials in the components of \mathbf{v} in virtue of equations (7), (33), and (35). Inasmuch as the unperturbed velocity distribution is isotropic, integrations over direction in the velocity space are readily performed, and the integrations over the velocity space reduce to the evaluation of the even moments

$$G(r, \mu) = m_* \int |\mathbf{v}|^\mu f_0(\mathbf{x}, \mathbf{v}) d\mathbf{v} \quad (\mu = 0, 2, \dots) \quad (\text{E1})$$

of the velocity distribution. It is a consequence of the collisionless Boltzmann equation that the moments satisfy the hierarchy of ordinary differential equations

$$\frac{d}{dr} G(r, \mu) = -(\mu + 1) G(r, \mu - 2) \frac{dV_0}{dr} \quad (\mu = 2, 4, \dots). \quad (\text{E2})$$

Thus, in the evaluation of the integrals on the right-hand sides of equations (B1) and (D6), we calculate the moments $G(r, \mu)$ of the velocity distribution by integrating equations (E2) with the boundary conditions $G(r, \mu) = 0$ at $r = R$, where R is the radius of the unperturbed system. We note in this connection that the zeroth moment of the velocity distribution is $G(r, 0) = m_* \int f_0(\mathbf{x}, \mathbf{v}) d\mathbf{v} = \rho_0(r)$.

The evaluation of the moments of the velocity distribution reduces the integrands on the right-hand sides of equations (B1) and (D6) to functions of the radial coordinate $r = |\mathbf{x}|$. Accordingly, these integrations over the configuration space reduce to quadratures over the radial coordinate r . Those quadratures can be reformulated in terms of the integration of associated ordinary differential equations with appropriate boundary conditions. The integration of these equations can be carried out simultaneously with the integration of equations (E2) described above. In this work, those integrations have been carried out with the aid of a standard Runge-Kutta integrator of the fourth order (Press et al. 1986).

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